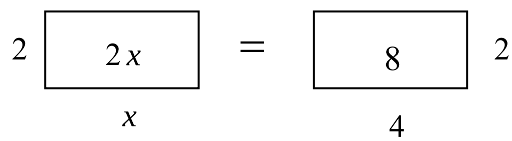
Visual Algebra Through Symmetry

George Mondras
Volunteer Mathematics Tutor
Ventura Community College, Ca

Abstract: We document Visual Algebra through Symmetry and reintroduce it into the mathematics pedagogy in modern notation. Reformulating an equation to obtain symmetry requires less symbolic logic, fewer arithmetic operations, activates the visual portion of mathematics intuition, and activates the student's most primitive concepts. Teaching both the axiomatic and visual methods in parallel would create a more powerful pedagogy that more closely corresponds to the mathematical abilities of the typical class in algebra.

Visual Algebra Through Symmetry is a method that was prevalent at the dawn of the mathematical sciences. At the University of Alexandria, 310 B.C.E, Euclid's students used symmetry in the solution of equations. For the equation: 2x = 8, each term in the equation was depicted as an area and a geometric diagram was constructed. Considering Euclid's fourth axiom, "Things which coincide with one another are equal to one another" is compelling evidence that algebra was geometric in that era. A student would conclude that equality requires that the two rectangular areas are equal; therefore *x* must coincide with 4.



Visual algebra is geometric algebra with symmetry substituted for geometry. The basis of the visual method is the reflective symmetry of the human form, a natural concept that implies a one-to-one matching of parts, is an organic part of human understanding, is ingrained very early in life and is independent of one's intrinsic mathematical ability. Symmetry in algebra is produced when an equation is reformulated such that the form of the equation is the same on both sides of the equality. The visual method does not solve for the variable, but applies the axioms to reformulate an equation to obtain symmetry. The solution is then found implicitly by a one-to-one matching of terms. Reformulation of an equation requires less symbolic logic, fewer arithmetic operations and activates the student's most primitive concepts.

Define Visual method:

Apply the axioms to one side of an equation to reformulate and achieve symmetry. Solve for the variable implicitly using a one-to-one matching of terms. Check any modified expression.

Visual Mantra: ***Rewrite the equation until both sides look the same.***

$ax + b = c$ Apply the axioms to one side of the equation to achieve final symmetry

$ax + b = a(\frac{c-b}{a}) + b$ Translational symmetry; Variable = Value by one-to-one matching

$ax + b = b + (\frac{c-b}{a})\text{a}$ Reflective symmetry is a possible reformulation, but not likely.

Define Axiomatic method:

Apply the axioms to both sides of an equation to simplify and reduce to an equivalent equation. Solve for the variable explicitly and check the solution in the original equation.

Axiomatic Mantra: ***Whatever you do to one side of an equation you must do to the other.***

$ax + b = c$ Apply the axioms to both sides of the equation to achieve final symmetry

$$x = \frac{c-b}{a}$$ Variable = Value by reflective symmetry

Note: The visual and axiomatic methods have an inverse relationship to each other in the sense that both achieve the unique solution through symmetry, one by axiomatic reformulation to translational symmetry and the other by axiomatic reduction to reflective symmetry.

Section 1, Linear Equations:

Linear equations are particularly well suited for a solution by the visual method. The method enables the student to determine by inspection the terms needed to obtain symmetry and note that the visual method requires no abstract symbolic logic. Generally, the abstract portion of an equation is kept unmodified and the real portion is reformulated such that the form of the equation is the same on both sides of the equality.

Key: LHS/RHS = Left/Right Hand Side. Group unmatched terms, Multiply by 1 = *1

L 1) Visual solution: Abstract equation; RHS: Reformulate to obtain symmetry.

$3x+5=11$ ← RHS: Add $0=-5+5$ and group the unmatched terms: (11 – 5)

$3x+5=(11-5)+5$ ← Multiply the term, (11 – 5) by $1=\frac{3}{3}$ or by $1=3(\frac{1}{3})$

$3x+5=3(\frac{11-5}{3})+5$ → $x=(\frac{11-5}{3})=2$, from the one-to-one matching of terms

Check: Final RHS = Original RHS? Does: $3(\frac{11-5}{3})+5=11$? Yes.

$3x+5=3(2)+5$ ← Translational symmetry; the more likely reformulation

$3x+5=5+(2)3$ ← Reflective symmetry; the less likely reformulation

$x=2$ ← Reflective symmetry; the axiomatic solution

3 [3x] + 5 = 3 [6] + 5
x 2

← Geometric Solution: 3x + 5 = 6 + 5
Euclid's 4th Axiom, area equality requires that x must coincide with 2.

L 1) Visual Alternative:

$3x+5=11$ ← Formulate and add a symmetrical equation

$3(\frac{1}{3})(11-5)=3x$ ← $x=(\frac{1}{3})(11-5)=2$ From one-to-one matching

$3x+11=3x+11$ ← Symmetry verifies that $x=2$, no additional check is needed.

L 2) A salesperson earns a salary of \$125 per week plus a commission of \$2.40 for each item sold. Find the number of items sold if the earnings were \$528.20 in one week.

$2.40n+125=528.20$ ← RHS: Add 0, group unmatched terms, $*1=2.40(\frac{1}{2.40})$

$2.40n+125=2.40(\frac{1}{2.40})(528.20-125)+125$ ← Evaluate *n:* $\frac{1}{2.40}(528.20-125)=168$

$2.40n+125=2.40(168)+125$ ← n = 168, items sold, from one-to-one matching

L 3) The cost to rent an automobile is \$37 dollars a day plus 21 cents per mile. If the final bill is \$61.15, how many miles were driven?

$0.21m+37=61.15$ ← RHS: Add 0, group unmatched terms, $*1=0.21(\frac{1}{0.21})$

$0.21m+37=0.21(\frac{1}{0.21})(61.15-37)+37$ ← Evaluate *m:* $\frac{1}{0.21}(61.15-37)=115$

$0.21m+37=0.21(115)+37$ m = 115 miles were driven, from one-to-one matching

Section 2, Systems of equations:

In systems of linear equations, where the variables have a tangible meaning, the equations will be reformulated to obtain symmetry or to contain a collinear equation. Examples: SE 1 to SE 10 introduces problem solving in two variables. The visual method will be used to obtain the solutions; current textbooks would use the axiomatic method exclusively.

SE 1) John is five years older than Mary. Two years ago, John was twice as old as Mary. What are their ages today? Reformulate each equation to obtain translational symmetry

$$J = 5 + M \rightarrow J - 2 = 5 + M - 2 + 2 - 2 \rightarrow J - 2 = 7 + M - 4$$
$$J - 2 = 2(M - 2) \rightarrow J - 2 = M - 2 + M - 2 \rightarrow J - 2 = M + M - 4$$

Mary's age is 7 years old, from one-to-one matching; John's age is 5 + 7 or 12 years old.

SE 2) There are two supplementary angles in which one angle is 12 degrees less than 3 times the other. What is the measure of the angles? Reformulate each equation to obtain translational symmetry.

$$x + y = 180 \rightarrow x + y = 192 - 12 \rightarrow x + y = 4(48) - 12$$
$$y = 3x - 12 \rightarrow x + y = x + 3x - 12 \rightarrow x + y = 4x \quad - 12$$
$x = 48°$ From one-to-one matching, $y = 180 - 48 = 132°$

SE 3) Cool Mitts, Inc., sold 20 pairs of gloves. Plain leather gloves sold for \$24.95 per pair and Gold-braided gloves sold for \$37.50 per pair. The company took in \$687.25. How many of each kind were sold?
Reformulate to contain a collinear equation with ratio = 1/24.95; 20(24.95) = 499.00

$$p + g = 20 \rightarrow p + g = 20$$
$$24.95p + 37.50g = 687.25 \rightarrow 24.95p + 24.95g + 12.55g = 499.00 + 188.25$$
$$g = \frac{188.25}{12.55} \rightarrow g = 15 \quad \text{Gold - braided gloves}$$
$$p = 20 - 15 \rightarrow p = 5 \quad \text{Plain leather gloves}$$

SE 4) Anti-freeze solution A is 2% methanol, and solution B is 6% methanol. Auto-Parts Inc., wants to mix the two to get 60 liters of a solution that is 3.2% methanol. How many liters of each solution is required?
Reformulate to contain a collinear equation with ratio = 1/2%

$$A + B = 60 \rightarrow A + B = 60$$
$$2\%A + 6\%B = 3.2\%(60) \rightarrow 2\%A + 2\%B + 4\%B = 2\%(60) + 1.2\%(60)$$
$$B = \frac{1.2\%(60)}{4\%} = 18 \text{ liters of 6\% methanol}$$
$$A + 18 = 60 \rightarrow 42 + 18 = 60 \rightarrow A = 42 \text{ liters of 2\% methanol}$$

Section 2, Systems of equations: (Continued)

SE 5) A $4800 investments in two corporate bonds earn $412 in interest the first year. The bonds pay interest of 8% and 9% per annum. Find the amount invested at each rate of interest.
Reformulate to contain a collinear equation with ratio = 1/0.08; 0.08(4800) = 384

$$E + N = 4800 \quad \rightarrow \quad E + N = 4800$$
$$0.08E + 0.09N = 412 \quad \rightarrow \quad 0.08E + 0.08N + 0.01N = 384 + 28$$

$N = \dfrac{28}{0.01}$ = $2800 at 9% interest; E = $4800 – $2800 = $2000 at 8% interest

SE 6) The ground floor of the Empire State building in New York is a rectangle. The perimeter of the building is 860 ft. The width is 100 ft less than the length. Find the length and width. Reformulate to contain a collinear equation with ratio = 2/1

$$2l + 2w = 860 \rightarrow 2l + 2w = 660 + 200 \rightarrow 2l + 2w = 4(165) + 200$$
$$l = 100 + w \rightarrow l + w = 100 + w + w \rightarrow l + w = 2w + 100$$

w = 165 ft from one-to-one matching. l = 100 + 165 = 265 ft

SE 7) A train leaves Boston traveling west at a speed of 30 km/h. Two hours later, another train leaves Boston traveling in the same direction on a parallel track at 45km/h. At what elapsed time will the faster train overtake the slower train?

$$Train\,A: \text{Distance} = 30(t+2) \rightarrow \text{D} = 30t + 60 \rightarrow \text{D} = 30t + 15(4)$$
$$Train\,B: \text{Distance} = 45t \rightarrow \text{D} = 30t + 15t \rightarrow \text{D} = 30t + 15t$$

t = 4 hours, from one-to-one matching

SE 8) A powered catamaran took 4 hours to make a trip downstream; the return trip took 5 hours. There was a constant 6-mph current. Find the speed of the catamaran in still water.

$$\text{Distance} = (v+6)4 \rightarrow \text{D} = 4v + 24 \rightarrow \text{D} = 4v - 30 + 54$$
$$\text{Distance} = (v-6)5 \rightarrow \text{D} = 5v - 30 \rightarrow \text{D} = 4v - 30 + v$$

v = 54 mph in still water, from one-to-one matching

Section 2, Systems of equations: (Continued)

In systems of abstract linear equations, one equation is kept unmodified while the other equation is reformulated to obtain the matching terms. The procedure eliminates the entire unmodified equation and yields a solution to the remaining variable.

SE 9) Visual solution, system of two equations: Reformulate *B* to eliminate *A*

$$A: 11x - 13y = 17 \quad \rightarrow \quad 11x - 13y = 17$$

$$B: 7x + 5y = 23 \rightarrow (\frac{11}{7}B): 11x - 13y + 13y + \frac{55}{7}y = 17 - 17 + \frac{253}{7} \quad \leftarrow (\frac{11}{7} * B, Add\ 0,\ 2\ places)$$

$$17 + (\frac{7(13) + 55}{7})y = 17 + (\frac{7(-17) + 253}{7}) \rightarrow \frac{146}{7}y = \frac{134}{7} \rightarrow y = \frac{134}{146}\ [y = \frac{67}{73}]$$

$$B: 7x + 5(\frac{67}{73}) = 23(\frac{73}{73}) \rightarrow x = \frac{23(73) - 5(67)}{7(73)} \quad [x = \frac{192}{73}]$$

$$Check, A: 11(\frac{192}{73}) - 13(\frac{67}{73}) = 17? \rightarrow \frac{2112}{73} - \frac{871}{73} = 17? \rightarrow \frac{1241}{73} = 17? \quad 17 = 17$$

$$Check, B: 7(\frac{192}{73}) + 5(\frac{67}{73}) = 23? \rightarrow \frac{1344}{73} + \frac{335}{73} = 23? \rightarrow \frac{1679}{73} = 23? \quad 23 = 23$$

SE 10) Visual solution, system of two equations with fractions as coefficients:

$$A: \frac{1}{11}x - \frac{1}{13}y = 17 \quad \rightarrow \quad \frac{1}{11}x - \frac{1}{13}y = 17$$

$$B: \frac{1}{7}x + \frac{1}{5}y = 23 \rightarrow (\frac{7}{11}B): \frac{1}{11}x - \frac{1}{13}y + \frac{1}{13}y + \frac{7}{55}y = 17 - 17 + \frac{161}{11} \quad \leftarrow \quad (\frac{7}{11} * B, Add\ 0,\ 2\ places)$$

$$17 + (\frac{55 + 13(7)}{13(55)})y = 17 + (\frac{11(-17) + 161}{11}) \rightarrow \frac{146}{13(55)}y = \frac{-26}{11} \rightarrow y = -\frac{13(55)(26)}{146(11)}\ [y = -\frac{845}{73}]$$

$$B: \frac{1}{7}x + \frac{1}{5}(-\frac{845}{73}) = 23(\frac{73}{73}) \rightarrow \frac{1}{7}x - \frac{169}{73} = \frac{1679}{73} \rightarrow x = \frac{7(1679 + 169)}{73} \quad [x = \frac{12936}{73}]$$

$$Check, A: \frac{1}{11}(\frac{12936}{73}) - \frac{1}{13}(-\frac{845}{73})y = 17? \rightarrow \frac{1176}{73} + \frac{65}{73}y = 17? \rightarrow \frac{1241}{73} = 17? \quad 17 = 17$$

$$Check, B: \frac{1}{7}(\frac{12936}{73}) + \frac{1}{5}(-\frac{845}{73})y = 23? \rightarrow \frac{1848}{73} - \frac{169}{73}y = 23? \rightarrow \frac{1679}{73} = 23? \quad 23 = 23$$

Section 3, Linear Inequalities:

Linear inequalities can be reformulated by keeping the abstract portion of the inequality unmodified. Add the equivalent of zeros to obtain the matching terms, and then multiply by the equivalent of one to obtain a matching coefficient. If the coefficient of the variable term is negative, the sense of the inequality must be reversed at the final step.

LI 1) Visual solution: RHS: Add 0, Group unmatched terms, multiply $*1 = -3(\frac{1}{-3})$

$-3x + 2 < 8 \rightarrow -3x + 2 < -3(\frac{+1}{-3})(8-2) + 2 \rightarrow x > \frac{+1}{-3}(8-2) = -2$

$-3x + 2 < -3(-2) + 2 \rightarrow$ Reverse the sense of the inequality on the last step

$x > -2$, from one-to-one matching

LI 2) Visual solution: LHS, RHS: Add 0, group unmatched terms, multiply $*1 = \frac{-2}{-2}$

$$-3 \le -2x + 4 \le 10$$
$$4 - 4 - 3 \le -2x + 4 \le 10 + 4 - 4$$
$$-7 + 4 \le -2x + 4 \le 6 + 4$$
$$\frac{-2}{-2}(-7) + 4 \le -2x + 4 \le \frac{-2}{-2}(6) + 4$$
$$-2(\frac{-7}{-2}) + 4 \le -2x + 4 \le -2(-3) + 4$$

Reverse the sense of the inequality. Range: $-3 \le x \le 3.5$ From one-to-one matching

LI 3) Visual solution:

$-3(x+8) - 5x < 4(x-9) + 27$ (*)$\rightarrow$ $-3x - 24 - 5x < 4x - 36 + 27$ (+)$\rightarrow$ $-8x - 24 < 4x - 9 \downarrow$

$-8x - 24 < 4x - 9 \leftarrow$ Formulate and add a symmetrical equation

$12x = -15$ $\quad x > -15 \div 12 \quad x > -1.25$ Reverse sense of the inequality

$4x - 24 = 4x - 24$ Symmetry verifies that $x > -1.25$

LI 4) Visual solution:

$4x - 2 \le x + 1 \le 3x + 4 \leftarrow$ LHS and RHS: Reformulate to match middle term

$3x - 3 + x + 1 \le x + 1 \le 2x + 3 + x + 1 \rightarrow 3x - 3 \le 0 \le 2x + 3 \leftarrow$ Cancel term $(x + 1)$, 3 places

$3x - 3 \le 0 \rightarrow 3x - 3 \le 3(\frac{3}{3}) - 3 \rightarrow x \le 1$ From one-to-one matching

$0 \le 2x + 3 \rightarrow +2(\frac{-3}{+2}) + 3 \le 2x + 3 \rightarrow -\frac{3}{2} \le x$ From one-to-one matching

$-\frac{3}{2} \le x \le 1$ Range of the variable

Section 4, Rational equations:

Rational equations are particularly difficult to comprehend precisely because they contain abstract fractions. The visual solution generally keeps the more difficult side of the equation unmodified and rewrites the less difficult side to obtain symmetry. In some cases, partial symmetry is created followed by division. Division generally produces remainders on both sides of an equation that must be equated repeatedly to reduce a rational equation to a linear form.

R 1) Visual solution: create partial symmetry to avoid the multiplications.

$$\frac{2x-7}{2x+11}=\frac{11}{29} \quad (+0)\rightarrow \quad \frac{2x+11-11-7}{2x+11}=\frac{29-29+11}{29} \quad (\div)\rightarrow \; 1+\frac{-18}{2x+11}=1+\frac{-18}{29}$$

$2x+11=29$ ← Equate denominators. Formulate and add a symmetrical equation

$2(9)=2x \quad \rightarrow \quad x=9$, from one-to-one matching

$2x+29=2x+29$ ← Symmetry verifies that $x = 9$, no additional check is needed.

R 2) Visual solution: create partial symmetry to avoid the binomial multiplications.

$$\frac{x+3}{x+5}=\frac{2x-7}{2x+11} \quad (+0)\rightarrow \quad \frac{x+5-5+3}{x+5}=\frac{2x+11-11-7}{2x+11} \quad (\div)\rightarrow \; 1+\frac{-2}{(x+5)}(\frac{9}{9})=1+\frac{-18}{2x+11}$$

$9x+45=2x+11$ ← Equate denominators. Formulate and add a symmetrical equation

$-7x \quad =-7(\frac{34}{-7})$ → $x=-\frac{34}{7}$, from one-to-one matching

$2x+45=2x+45$ ← Symmetry verifies that $x=-\frac{34}{7}$, no additional check is needed.

R 3) Visual solution: create partial symmetry to avoid the binomial multiplications.

$$\frac{2x-7}{2x+11}=\frac{x+3}{x+8} \quad (+0)\rightarrow \quad \frac{2x+11-11-7}{2x+11}=\frac{x+8-8+3}{x+8} \quad (\div)\rightarrow \; 1+\frac{-18}{2x+11}=1+\frac{-5}{x+8}$$

$(\frac{5}{5})\frac{-18}{(2x+11)}=(\frac{18}{18})\frac{-5}{(x+8)}$ ← Multiply by $*1 = \frac{5}{5} = \frac{18}{18}$

$10x+55=18x+144$ ← Equate denominators. Formulate and add a symmetrical equation

$8x=8(\frac{1}{8})(55-144) \qquad x=(\frac{1}{8})(55-144)=\frac{-89}{8} \; = -11.125$, from one-to-one matching

$18x+55=18x+55$ ← Symmetry verifies that $x = -11.125$, no additional check is needed.

Section 4, Rational equations: (Continued)

R 4) Visual solution: rational equation containing multiple fractions

$\frac{x+5}{2}+\frac{1}{2}=2x-\frac{x-3}{8}$ → In symbols → LHS = RHS → LHS – RHS + RHS = RHS

$\frac{x+5+1}{2}-2x+\frac{x-3}{8}$ ← Isolate unmatched terms: LHS – RHS, multiply by 1 = $\frac{4}{4}$ = $\frac{8}{8}$

$\frac{x+6}{2}(\frac{4}{4})-2x(\frac{8}{8})+\frac{x-3}{8}=\frac{4x+24}{8}+\frac{-16x}{8}+\frac{x-3}{8}=\frac{-11x+21}{8}$ ← Evaluate

$\frac{-11x+21}{8}+2x-\frac{x-3}{8}=2x-\frac{x-3}{8}$ ← Final equation

Unmatched term (numerator) must equal zero; reformulate to contain an inverse coefficient

$-11x+21=0 \ \Rightarrow \ -11x+11(\frac{21}{11})=0, \quad x=\frac{21}{11}$, From one-to-one matching

This unique value produces symmetry in the final equation, $2x-\frac{x-3}{8}=2x-\frac{x-3}{8}$ and thereby verifies that, $x=\frac{21}{11}$. LHS expression check, use $x = 0$ to check. Original LHS = Final LHS?

Original LHS: $\frac{5}{2}+\frac{1}{2}=\frac{6}{2}=3$. Final LHS: $\frac{21}{8}-\frac{-3}{8}=\frac{24}{8}=3$

R 5) Visual solution: rational equation with multiple fractions.

$\frac{x-4}{3x}+\frac{x-8}{5x}=\frac{-16}{x}$ → In symbols → LHS = RHS → LHS = LHS – LHS + RHS

$-\frac{x-4}{3x}-\frac{x-8}{5x}+\frac{-16}{x}$ ← Isolate unmatched terms: – LHS + RHS and multiply by 1 = $\frac{5}{5}=\frac{3}{3}=\frac{15}{15}$

$\frac{5}{5}(-\frac{x-4}{3x})+\frac{3}{3}(-\frac{x-8}{5x})+\frac{15}{15}(\frac{-16}{x}) = \frac{-5x+20}{5(3x)}+\frac{-3x+24}{3(5x)}+\frac{-240}{15x}=\frac{-8x-196}{15x}$ ← Evaluate

$\frac{x-4}{3x}+\frac{x-8}{5x}=\frac{x-4}{3x}+\frac{x-8}{5x}+\frac{-8x-196}{15x}$ ← Final equation

Unmatched term (numerator) must equal zero, reformulate to contain an inverse coefficient.

$-8x-196=0$ → $-8x+8(\frac{-196}{+8})=0$ → $x=-\frac{49}{2}$ from one-to-one matching.

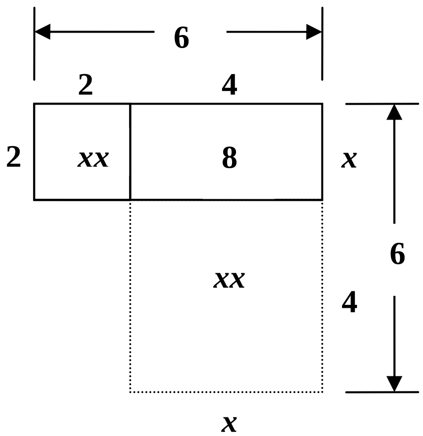
This unique value produces symmetry in the final equation, $\frac{x-4}{3x}+\frac{x-8}{5x}=\frac{x-4}{3x}+\frac{x-8}{5x}$ and thereby verifies that $x=-\frac{49}{2}$. Check RHS expression; does the Final RHS = Original RHS?

$\frac{x-4}{3x}+\frac{x-8}{5x}-\frac{x-4}{3x}-\frac{x-8}{5x}+\frac{-16}{x}=\frac{-16}{x}$? → LHS – LHS + $\frac{-16}{x}=\frac{-16}{x}$? → $\frac{-16}{x}=\frac{-16}{x}$

Note: In **R** 4) and **R** 5), the axiomatic method (multiply both sides by 8 and 15x, etc.) requires 26 and 24 arithmetic operations, including the check, versus 11 and 9 for the visual method.

Section 5, Quadratic equations:

In quadratic equations, completing-the-square is currently in vogue as the only general solution of these equations. A geometric-numerical method, formulated at the dawn of the mathematical sciences, is also a general solution. At the University of Alexandria, 310 BC, Euclid's students were equipped with a straightedge and compass and could construct the geometric diagrams to scale. One possible diagram for the equation, *xx – 6x + 8 = 0*, is constructed by depicting the term 6x as an area. A rectangle is drawn with length 6 units and width *x* units. The area of the rectangle must equal *xx* + 8 square units. Noting that 2 * 4 = 8, and that 2 + 4 = 6, area *xx* is drawn as a 2 * 2 square; and the lower 4 * 4 square, drawn dashed, completes the diagram. Only the diagrams of quadratic equations with positive roots were constructed; negative numbers and exponent notation were invented in a later era.

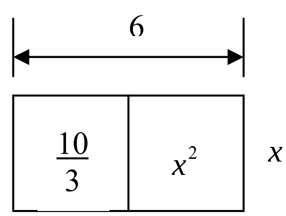


6x + 8 = 0
8 = 6x
*xx + (2 * 4) = (2 + 4)x*
Roots = [2, 4]

Geometric-Numerical Method:

$$\text{Example}: f(x) = 3x^2 - 18x + 10 = 0 \ \text{Find the roots.}$$

$$3x^2 + 10 = 18x \ \rightarrow \ x^2 + \frac{10}{3} = 6x \quad \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$$



Assume the equation is a trinomial square, that is, area $\frac{10}{3}$ = area x^2; the *x*-coordinate at the vertex of a trinomial square is: $x = \frac{6}{2} = 3.$ Does: $3^2 + \frac{10}{3} = 6(3)$? No.

The roots will deviate from the roots of a trinomial square by an amount:

$$e^2 = 18 - 9 - \frac{10}{3} \ \rightarrow \ e^2 = \frac{3(9) - 10}{3} \ \rightarrow \ e^2 = \frac{17}{3}$$

The roots are therefore: $x = 3 \pm \sqrt{\frac{17}{3}}$ *or* $x = 3 \pm \frac{\sqrt{51}}{3}$

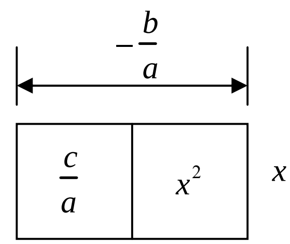
A generalized derivation follows. Those students that have great difficulty with mathematical abstraction may find the above derivation much easier to comprehend.

Section 5, Quadratic equations: (Continued)

Generalized Geometric-Numerical method:

General equation: $f(x) = ax^2 + bx + c = 0$

$$x^2 + \frac{c}{a} = -\frac{b}{a}x \quad \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$$



The diagram is symmetrical if the equation is a trinomial square, that is, area $\frac{c}{a}$ = area x^2

The redundant roots are: $x = [-\frac{b}{2a}, -\frac{b}{2a}]$; assuming that every quadratic equation has these roots would produce an error; $|e^2| \geq 0$. The deviation from a trinomial square can be computed:

$$e^2 = -x^2 - \frac{b}{a}x - \frac{c}{a} \quad \Rightarrow \quad e^2 = -(-\frac{b}{2a})^2 - \frac{b}{a}(-\frac{b}{2a}) - \frac{c}{a} \quad \Rightarrow \quad e^2 = -\frac{b^2}{4a^2} + (\frac{b^2}{2a^2}) - \frac{c}{a}$$

$$e^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \Rightarrow \quad e^2 = \frac{(\frac{b}{2})^2}{a^2} - \frac{ac}{a^2} \quad \Rightarrow \quad e^2 = \frac{(\frac{b}{2})^2 - ac}{a^2} \qquad Also: f(-\frac{b}{2a}) = \frac{-e^2}{a}$$

The roots of a quadratic equation are: $x = \frac{-\frac{b}{2}}{a} \pm \frac{\sqrt{e^2}}{a}$, and the vertex is: $v = (-\frac{b}{2a}, \frac{-e^2}{a})$

Redefine e^2 as the numerator: $(\frac{b}{2})^2 - ac$, and express as a determinant to facilitate computation.

Example: $f(x) = 3x^2 - 18x + 10, \quad e^2 = \begin{vmatrix} -\frac{b}{2} & a \\ c & -\frac{b}{2} \end{vmatrix} = \begin{vmatrix} -\frac{-18}{2} & 3 \\ 10 & -\frac{-18}{2} \end{vmatrix} = \begin{vmatrix} 9 & 3 \\ 10 & 9 \end{vmatrix} = [81] - [30] = 51$

Roots: $x = \frac{9}{3} \pm \frac{\sqrt{51}}{3} \quad \Rightarrow \quad x = 3 \pm \frac{\sqrt{51}}{3}$ Vertex: $v = (3, \frac{-51}{3}) \quad \Rightarrow \quad v = (3, -17)$

The Geometric-Numerical method requires (6) arithmetic operations to compute the roots and vertex. Considering that three coordinates can be found quickly using only arithmetic, the method lends itself to graphing quadratic equations. In this example, the quadratic formula requires (9) arithmetic operations to compute the roots alone. The geometric-numerical method also calls into question the need to learn the factoring procedures to solve quadratic equations.

General Quadratic Equation: $f(x) = ax^2 + bx + c$

Symmetrical form: $ax(x + \frac{b}{a}) + c = a[-\frac{b}{2a} \pm \frac{\sqrt{e^2}}{a}]([-\frac{b}{2a} \pm \frac{\sqrt{e^2}}{a}] + \frac{b}{a}) + c$

Factored Form: $(x + \frac{b}{2a} + \frac{\sqrt{e^2}}{a})(x + \frac{b}{2a} - \frac{\sqrt{e^2}}{a}) = 0$

Section 5, Quadratic equations: (Continued)

Geometric-Numerical solution of quadratic equations:

$$e^2 = \begin{vmatrix} -\frac{b}{2} & a \\ c & -\frac{b}{2} \end{vmatrix}, \quad Roots: x = \frac{-\frac{b}{2}}{a} \pm \frac{\sqrt{e^2}}{a}, \quad Vertex: v = \left(\frac{-\frac{b}{2}}{a}, \frac{-e^2}{a}\right)$$

The trinomial square deviation: e^2, determines the nature of the solutions

Q5.1) $9x^2 - 24x + 16 = 0 \qquad e^2 = \begin{vmatrix} 12 & 9 \\ 16 & 12 \end{vmatrix} = [144] - [144] = 0 \qquad e^2 = 0$

e^2 is zero, equation is a trinomial square. Roots: $x = \frac{4}{3} \pm 0$, $Vertex:\ v = (\frac{4}{3}, 0)$

Q5.2) $2x^2 - 6x - 7 = 0 \qquad e^2 = \begin{vmatrix} 3 & 2 \\ -7 & 3 \end{vmatrix} = [9] - [-14] = 23 \qquad e^2 = 23$

e^2 is positive, roots are irrational. Roots: $: x = \frac{3}{2} \pm \frac{\sqrt{23}}{2}$, $Vertex: v = (\frac{3}{2}, \frac{-23}{2})$

Q5.3) $-3x^2 - x + 2 = 0 \quad e^2 = \begin{vmatrix} \frac{1}{2} & -3 \\ 2 & \frac{1}{2} \end{vmatrix} = [\frac{1}{4}] - [-6] \qquad e^2 = \frac{4(6)+1}{4} \qquad e^2 = \frac{25}{4}$

e^2 is a positive square, roots are rational. Roots: $x = -\frac{1}{6} \pm \frac{5}{6}$, $Vertex: v = (-\frac{1}{6}, \frac{25}{12})$

Q5.4) $3x^2 - 6x + 13 = 0 \qquad e^2 = \begin{vmatrix} 3 & 3 \\ 13 & 3 \end{vmatrix} = [9] - [39] = -30 \qquad e^2 = -30$

e^2 is negative, roots are complex. Roots: $x = 1 \pm \frac{\sqrt{-30}}{3}$, $Vertex: v = (1, 10)$

Or, roots: $x = 1 \pm \frac{\sqrt{30}}{3} i$, $where\ i = \sqrt{-1}$

Section 6, Exponential equations:

Exponential equations with variables or irrationals as exponents can be reformulated to obtain symmetry by using the change of base formula in logarithms. Symmetrical exponential equations eliminate the need to take the logarithm of both sides of an equation.

E 1) Visual solution: an equation containing a variable exponent.

$$24 = 5(2)^x \quad \Rightarrow \quad 5(\frac{24}{5}) = 5(2)^x \quad \Rightarrow \quad 5(4.8) = 5(2)^x$$

$$5(2)^{\frac{\log(4.8)}{\log(2)}} = 5(2)^x \qquad 5(2)^{\frac{Ln(4.8)}{Ln(2)}} = 5(2)^x$$

$$x = \frac{Log(4.8)}{Log(2)} \text{ or } x = \frac{Ln(4.8)}{Ln(2)} \text{ from one-to-one matching.}$$

E 2) Visual solution: an equation containing an irrational exponent.

$$3x^{\pi} - 7 = 4 \Rightarrow 3x^{\pi} - 7 = 3(\frac{4+7}{3}) - 7 \Rightarrow 3x^{\pi} - 7 = 3x^{\frac{Ln(\frac{11}{3})}{Ln(x)}} - 7$$

$$\pi = \frac{Ln(\frac{11}{3})}{Ln(x)} \text{ From one-to-one matching. } Ln(x) = \frac{Ln(\frac{11}{3})}{\pi} \quad \Rightarrow \quad x = 1.51221$$

E 3) Visual solution: an equation containing a variable exponent and base.

$$y = x^x \Rightarrow x^{\frac{\ln y}{\ln x}} = x^x \Rightarrow \frac{\ln y}{\ln x} = x, \text{ From one-to-one matching}$$

$$\ln y = x \ln x \quad \Rightarrow \quad \text{limit as } x \to 0; \quad \ln y = 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad \therefore 0^0 = 1$$

Note: this statement, or its equivalent, appears in all of the algebra textbooks in print.

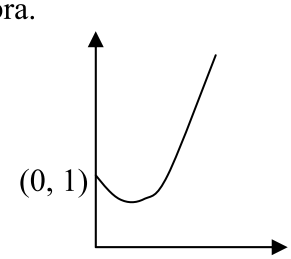
For any real number: a, $a \neq 0$, $a^0 = 1$.

Any nonzero number raised to the zero power is 1.

This is the eternal textbook error as shown in example **E** 3). While the error is minor, the statement is false and does not belong in a textbook on algebra.

Calculator check: y = $0.000001^{0.000001}$ = 0.99999

The graph of $y = x^x$ will show the limit (0, 1) → → →



Conclusion:

The experimental evidence that verifies that all human abilities are normally distributed is substantial. The center point mean, of a normal distribution, implies that one-half of the student populations will be below average in mathematical ability. Given a birth rate of 4.4 million per year (US, 2007) or students per grade, approximately 29 million students are below average in mathematical ability. This is the existential condition that a teacher is confronted with in K-12.

Every student, to some degree, has a measure of mathematical ability that depends on the amount that one accumulates the mental representation of mathematical objects whose properties are reproducible. Intuition is the faculty by which one can consider or examine the mathematical objects that are stored in a mental set of neurons. When contemplating a problem, internal mathematical intuition is the faculty of browsing in one's neuron library until a new insight or connection between the objects is found. When contemplating a diagram that has a quantitative connotation, visual mathematical intuition is the faculty that enables one to perceive the mathematical truth revealed in the diagram.

The axiomatic method has reigned too long as the exclusive basis of the mathematics pedagogy to the detriment of those students who lack the degree of intuition necessary to comprehend the concept of formal proof. By comparison, the reflective symmetry of the human form is a natural concept that is ingrained in everyone (**see Note***) and was once utilized in the early geometric development of algebra. The visual method is geometric algebra, with symmetry substituted for the geometry, and has been absent from the pedagogy for approximately two millennia. Given the existential conditions that exist in the typical classroom, there is a desperate need to re-introduce the method into the pedagogy.

Because visual learning is the dominant mode, visual mathematical intuition is also the dominant mode. These facts strongly suggest that teaching both the axiomatic and visual methods in parallel, both in the classroom and in the textbooks, would create a more powerful pedagogy that more closely corresponds to the mathematical abilities of the typical class in algebra. Moreover, the visual method may be the only means available to rescue those students whose circumstance places them at high risk of failure in abstract mathematics.

Note*: If these assertions are true, then visual algebra can be taught starting in the first grade. A special case of subtraction by addition enables the students who are learning integer addition to be learning subtraction by default. The special case is accurate **if and only if** a 2-digit minuend **mn** is in the **range**, **10 ≤ mn ≤ 19** and the subtrahend **s** is in the **range**, **0 ≤ s ≤ 9**. Differences are obtained by adding the minuend digits and the nine's complement of the subtrahend. The algorithm is: **mn** – **s** = **m** + **n** + (9 – **s**), where (9 – **s**) is the nine's complement (**9C**) of the subtrahend. The (**9C**) table for the base 10 digits is: (**0 + 9**), (**1+ 8**), (**2 + 7**), (**3 + 6**), (**4 + 5**). When the students have learned integer addition, they can solve simple algebraic equations. At first the students would be asked to identify the variable value of many equations that are in translational symmetry form, some examples are: 3x + 5 = 3(2) + 5, 2(x – 6) = 2(3 – 6), x + 2x + 5x = x + 2x + 5(4), 7[3x + 2] – 4 = 7[3(1) + 2] – 4; then later, equations that require simple operations to achieve symmetry: 3x + 5 = 3(1 + 1) + 5, 3x + 5 = 3(5 – 2) + 5, 7[3x + 2] = 7[3(3 + 2)], etc.
When the students acquire arithmetic and one-to-one matching skills, they would progress to solving simple linear equations, for example: *x + 5 = 11* → *x + 5 = 11 – 5 + 5* → *x + 5 = 6 + 5* → *x = 6*, from one-to-one matching. The subtraction in the equation could be done by any student skilled in integer addition, $11 - 5 = 1 + 1 + 4_5 = 6$, where 4 is the nine's complement of the subtrahend 5 (shown as a subscript). Accuracy of the nine's complement procedure is verified by the equation: 11 – 5 = (9 + 2) – (9 – 4) = 2 + 4 = 6. Other examples of the nine's complement procedure are: $11 - 2 = 1 + 1 + 7_2 = 9$, $13 - 7 = 1 + 3 + 2_7 = 6$ and $17 - 9 = 1 + 7 + 0_9 = 8$.
In applying the special case, a student's thinking may go something like this:

14	→	+ 5	13	→	+ 4	← Add the minuend digits: 1 + 4 and 1 + 3
$-\underline{9}$	→	$+\underline{0}$	$-\underline{7}$	→	$+\underline{2}$	← A number that added to 9, 7, that equals 9 is 0, 2, or use the 9C table
		+ 5			+ 6	← Difference obtained by the addition: 0 + 5 and 2 + 4

Teaching visual algebra, starting in the first grade, would familiarize the students with abstract mathematics and thereby eliminate the fear and anxieties experienced when textbook (axiomatic) algebra is encountered.

George Mondras is a Volunteer Mathematics Tutor at Ventura Community College, CA. Mr Mondras claims that, at 76, he is too old to claim credit for a manuscript. His motive for writing this article is to help the at-risk students by simplifying the mathematical procedures as much as possible.