

# Ratio and Proportion in Euclid

James J. Madden  
Louisiana State University

*Abstract: Modern conceptions of ratio and proportion are strongly influenced by Euclid's treatment in Book V of the Elements. Yet, there is a big difference between how we think of these ideas and how Euclid thought about them. We naturally attach measurements to the objects we compare, and because we do this we think of ratios as numbers. Euclid, in contrast, thought of a ratio as a direct comparison between things. For him, a ratio was not a number.*

## Introduction

The topic of ratio and proportion has been part of school mathematics since modern schooling began. It includes the advanced arithmetic and rudimentary algebra used in dealing with quantities  $x$  and  $y$  that are related by an equation of the form  $y = kx$ , where  $k$  is a constant, and it has numerous applications in science and commerce.

Much of what is taught in school comes from two sources. The arithmetical root is the ancient procedure known as the "Rule of Three", used for solving problems such as finding the price of one quantity of some commodity if the price of another quantity of the same commodity is known. This is the math that Lincoln was referring to when he wrote, in the brief autobiography that he prepared prior to the 1860 election: "We reached our new home [in Spencer County, Indiana] in my eighth year... It was a wild region, with many bears and other wild animals, still in the woods. There I grew up. There were some schools, so called; but no qualification was ever required of a teacher beyond "readin, writin, and cipherin" to the Rule of Three." In the surviving pages of his school notebook, Lincoln wrote out the solution to the following problem, which comes from Thomas Dilworth's *Schoolmaster's Assistant*: "If 3 ounces of silver cost 17 shillings what will 48 ounces cost?"

In this essay, we uncover the geometric roots of proportional reasoning. The earliest reference is Book V of Euclid's *Elements*, where Euclid develops the concepts of ratio and proportion for (unmeasured) geometric magnitudes. The words "ratio" and "proportion" are in fact derived from the Latin words that Cicero suggested as renderings for Euclid's words *logos* and *analogia*. Book V contains 18 definitions and 25 propositions on the theory of ratio and proportion. Heath, in his commentary, writes:

*The anonymous author of a scholium to Book V, who is perhaps Proclus, tells us that 'some say' this Book, containing the general theory of proportion which is equally applicable to geometry, arithmetic, music, and all mathematical science, 'is the discovery of Eudoxus, the teacher of Plato.'*

What is this theory, and how does it relate to the modern conceptual system of ratio

and proportion?

The chief concerns of Euclid's theory, as we shall see, are very different from those that arise when dealing with measured quantities. Euclid's theory, in fact, includes (implicitly) an analysis of the measurement process itself, and in this it goes far deeper than anything we find in typical schoolbook expositions of ratio and proportion. Understanding this could be very useful for teachers who are trying to help students develop clear ideas that link work with numbers to the things they measure.

### **Contrasts between Ancient and Modern Mathematics**

The mathematics of ancient Greece differs from modern mathematics. Number systems were more rudimentary, and numbers and numerical measurements played a very limited role in Euclidean geometry.

Aristotle, in his work *Categories* (written about 50 years before Euclid), classified quantity as "either discrete or continuous." The primary example of a discrete quantity was number, and number meant *a collection of units*. For the Greeks, the numbers were the positive integers—nothing more. They had no "real number system" and no "number line". Among the continuous quantities, on the other hand, one found the objects of geometry: segments, planar regions, and other things that Euclid later referred to as "magnitudes". They were not numbers, and had no numbers attached. Ratio, as treated in Book V, is a relationship between magnitudes. As such, it is an abstraction outside the realm of number.

A passage written by Newton in the 1660s shows that by then the European conception of number had been completely altered, turning the Greek conception on its head:

*By Number we understand, not so much a Multitude of Unities, as the abstracted Ratio of any Quantity, to another Quantity of the same Kind, which we take for Unity. And this is threefold; integer, fracted, and surd: An Integer, is what is measured by Unity; a Fraction, that which a submultiple Part of unity measures; and a Surd, to which Unity is incommensurable. (Universal Arithmetic, translated from the Latin by Ralphson, London 1769, page 2.)*

The new and broader conception of number predated Newton, of course, but this is not the place to explore its history. It suffices to note how fundamentally different the modern conception of number is from the ancient one. For us, the counting numbers are merely the most rudimentary level of a hierarchy of ever more general systems, culminating with the real numbers (or beyond). Our richer number system provides us with the machinery for treating ratio and proportion in a much different manner from the ancient Greeks.

A second major difference between ancient and modern mathematics concerns the

immediacy of the objects, as we see in the following passage from David Fowler's reconstruction of the ancient Greek mathematical mentality:

*Greek mathematicians seemed to confront directly the objects with which they were concerned: their geometry dealt with the features of geometrical thought experiments in which figures were drawn and manipulated, and their arithmetike concerned itself ultimately with the evident properties of numbered collections of objects. Unlike the mathematics of today, there was no elaborate conceptual machinery, other than natural language, interposed between the mathematician and his problem. Today we tend to turn our geometry into arithmetic, and our arithmetic into algebra so that, for example, while Elements I.47: "In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle" means literally to Euclid, that the square can be cut in two and manipulated into other squares. . . , the result is now usually interpreted as: " $p^2+q^2 = r^2$ ," where we now must explain just what the 'p's, 'q's and 'r's are and how they can be multiplied and added. To us, the literal squares have been replaced by some abstraction from an arithmetical analogy. (David Fowler. *The Mathematics Of Plato's Academy: A New Reconstruction. Second Edition.* Oxford: Clarendon Press, 1999, page 20.)*

The point that Fowler is making is not about a difference in depth or abstraction, but rather about how pervasively and automatically we moderns translate geometry into numerical and algebraic language and deal with geometric facts and ideas in these terms. We take for granted a much more immediate connection between numbers and things—a broad range of things—than the ancients ever conceived. For us, virtually everything has a numerical measure attached: distance, mass, time, price, academic performance. Universal quantification is a mark of the mentality our time. (One speculates that our obsession with attaching numbers to objects may be related to commerce. There are few things that we can obtain in the modern world that do not have a price attached. When we see something we want, the first thought we are likely to have is, "How much will it cost?")

We think of a ratio as a number obtained from other numbers by division. A proportion, for us, is a statement of equality between two "ratio-numbers". When we write a proportion such as  $a/b=c/d$ , the letters refer to numbers, the slashes are operations on numbers and the expressions on either side of the equals sign are numbers (or at least become numbers when the numerical values of the letters are fixed).

This was not the thought pattern of the ancient Greeks. When Euclid states that the ratio of  $A$  to  $B$  is the same as the ratio of  $C$  to  $D$ , the letters  $A$ ,  $B$ ,  $C$  and  $D$  do not refer to numbers at all, but to segments or polygonal regions or some such magnitudes. The ratio itself, according to Definition V.3, is just "a sort of relation in respect of size" between magnitudes. Like the definition of "point", this tells us little; the real meaning is found in the use of the term. It is in the rules for use that we find the

amazing conceptual depth of the theory.

The definition that determines how ratios are used is V.5. This tells us how to decide if two ratios are the same. The key idea is as follows. If we wish to compare two magnitudes, the first thing about them that we observe is their relative size. They may be the same size, or one may be smaller than the other. If one is smaller, we acquire more information by finding out how many copies of the smaller we can fit inside the larger. We can get even more information if we look at various multiples of the larger, and for each multiple, determine how many copies of the smaller fit inside. So, a ratio is implicitly a comparison of all the potential multiples of one magnitude to all the potential multiples of the other. (Two magnitudes are incommensurable exactly when no multiple of one is ever exactly equal to any multiple of the other.) To compare two ratios,  $A:B$  and  $C:D$ , then, we ought to be prepared to compare the array of all possible (whole-number) multiples of the first pair with the array of all possible (whole-number) multiples of the second. Suppose that we find that for each pair of positive integers  $(m, n)$ ,  $mA$  exceeds  $nB$  exactly when  $mC$  exceeds  $nD$ . This, according to Euclid's definition, is when we say the ratios are the same.

### **What properties do Euclidean magnitudes possess?**

As we have said, a ratio is a relationship between magnitudes. To understand Euclid's theory fully, then, we need to know what magnitudes are. Segments, polygonal regions in the plane, volumes and angles were clearly included, but beyond these kinds, we do not know what other things Euclid might have viewed as magnitudes. Fortunately, it is not hard to infer from Book V the properties that a class of things must have, and what operations we must be able to perform on them, in order to be able to apply the theory.

There are clear indications in his writings of the following:

A) Magnitudes are of several different kinds, e.g., segments, polygonal regions, volumes, angles—possibly weights and durations.

B) Given two magnitudes of the same kind, exactly one of the following is true: i) they are the same with respect to size (we say "equivalent"), ii) the first exceeds the second or iii) the second exceeds the first. (We call this the law of trichotomy for magnitudes.)

C) Magnitudes of the same kind may be added to one another—or a given magnitude may be added to itself one or more times—to yield a new magnitude of the same kind that is larger than any summand. No matter how the addition is performed, the outcome has the same size. Furthermore, given two magnitudes of the same kind but of different size, a part of the larger equivalent to the smaller may be removed, and no matter how this removal is done, the remainders are equivalent.

D) The relationships of equivalence and of excess are compatible with addition and subtraction in the sense that if equivalent magnitudes are added to (or taken from) each of two others, the resulting magnitudes will be in the same relation as the originals.

For Euclid, addition or subtraction of magnitudes was a concrete process. In the case of segments, addition and subtraction are described in Book I, Propositions 2 and 3. The addition of polygonal regions is treated in Book I beginning in the proof of Proposition 35: “Parallelograms which are on the same base and in the same parallels equal one another.” The discussion continues through the proof of the Pythagorean Theorem. As a matter of fact, Euclid’s proof of the Pythagorean Theorem is itself an explicit procedure for slicing up two square regions and rearranging the parts to make a third square region which is their sum.

In general, there may be numerous ways to add two magnitudes. For example, when two polygonal regions are added, they may be cut into pieces and reassembled in many different ways. Euclid took for granted that when addition of the same magnitudes is performed in two different ways, the results will always be equivalent, even if the relations between the assembled parts in each result are different.

Some Euclidean magnitudes can be multiplied, in a sense, but multiplication changes the kind. The product of two segments, for example, is a rectangle. This form of multiplication plays an important role in Book II, at the heart of the “geometric algebra” presented in that book. When multiplication of magnitudes is possible, multiplication of ratios can be defined. A simple form of this idea is the statement that if  $A$ ,  $B$  and  $C$  are segments and  $AC$  and  $BC$  are the rectangles with bases  $A$  and  $B$  and common height  $C$ , then  $(A:B)$  is the same as  $(AC:BC)$ .

## Conclusions

The most basic objects in Euclid’s theory of ratio and proportion are the so-called magnitudes. Segments, planar regions, volumes and angles are examples of magnitudes.

If any two magnitudes of the same kind are compared with respect to size, they will either be found to be of the same size, or one will be bigger. Magnitudes of the same kind may also be added to one another—often in numerous different ways. Yet no matter how magnitudes are added, the size of the result depends only on the sizes of the ingredients. A smaller magnitude may be removed from a larger one of the same kind, also with no ambiguity regarding the size of the result. Adding or subtracting the same magnitude to or from two others preserve the size relation; of the two that are increased or diminished by the same magnitude, the larger remains larger.

In speaking of the size of a magnitude, Euclid is *not* referring to a numerical measurement, but only to the ordering of the magnitudes of a given kind by size.

Today, we measure objects to find a number that indicates size with respect to a standard, but no such measurement is presupposed by Euclid's theory.

Given two magnitudes of the same kind, the ratio between them is a more precise indicator of relative size than the mere order (bigger/smaller). We may think of the ratio as the array of all comparisons of all possible multiples of the first magnitude with all possible multiples of the second. Thus, the ratio of  $A$  to  $B$  is characterized by the set of all pairs  $(m, n)$  such that  $mB$  is less than  $nA$  in size.

In forming a ratio, the terms must be like magnitudes because direct comparisons of magnitude to magnitude are necessary. But once a ratio between objects of one kind is made, it may be compared to a ratio between objects of another kind. The ability to compare ratios of one kind of magnitude with ratios of another kind is one of the most powerful and important aspects of Euclid's theory.

All the propositions that Euclid proved about ratios have analogues in the arithmetic of real numbers. We make the translation by selecting a standard unit magnitude  $U$  in each kind of magnitudes. In place of "the ratio of  $A$  to  $B$ ", we think of "the number  $a/b$ ", where  $a$  is the measure of  $A$  with respect to  $U$  and  $b$  is measure of  $B$  with respect to  $U$ .

### **Suggestions**

Modern treatments of ratio and proportion are arithmetic/algebraic in form. The student works with equations and numbers. The numbers have a link to things. But this link is not "in the arithmetic", and often not on paper during the work.

Experience suggests that it is often not in mind, either. Consider, for example, the following problem: "If 48 ounces of orange juice costs \$3.45, what should a quart (32 ounces) cost?" The arithmetic that gives the answer might start with the equation:

$$32/48 = x/3.45.$$

The equation is significant because the 48 is a number of ounces of orange juice with known value and the 32 is also a number of ounces of the same thing, but of unknown value. And the ratio of 32 ounces of orange juice to 48 ounces of orange juice is the same as the ratio of the unknown cost to the known cost of \$3.45. But the equation does not say this. It contains no hint of the origin of the numbers or their relationships to things. The equation, by itself, is just a statement about  $x$ .

The mathematics of Euclid is not a mathematics of numbers, but a mathematics of things. The symbols, relationships and manipulations have physical or geometric objects as their referents. You cannot work on this mathematics without knowing the objects that you are working with. A challenge for educators is to find a way of bringing this level of awareness of meaning into the modern, arithmetic/algebraic

treatment in an appropriate way.

*Acknowledgement.* This is an abridged version of a longer essay that was prepared with support from the Noyce Foundation and the University of Texas Dana Center as a contribution to a project on school mathematics headed by Dick Stanley of the University of California, Berkeley.

*Further reading.* Useful commentaries on Euclid but to address issues that are associated with teaching include Alexander John Ellis, "Euclid's conception of ratio and proportion," in *Algebra Identified with Geometry*, London: C. F. Hodgson & Sons 1874, and Augustus De Morgan, *The Connection of Number and Magnitude: An Attempt to Explain the Fifth Book of Euclid*. London: Taylor and Walton 1836. Reprints of these books are available on line.

*Mathematical connections.* The mathematician O. Hölder developed Euclid's theory into a rigorous theory of measurement that has important applications in the sciences. See Joel Michell and Catherine Ernst, "The Axioms of Quantity and the Theory of Measurement: Translated from Part I of Otto Hölder's German Text "Die Axiome der Quantität und die Lehre vom Mass," *Journal of Mathematical Psychology*, Volume 40, Issue 3, September 1996, Pages 235-252.

---

*James Madden obtained his Ph. D. in Mathematics from Wesleyan University in 1984. He is presently Professor of Mathematics at Louisiana State University (LSU) and codirector of the LSU Cain Center for STEM Literacy.*