

Greatest Common Factors and Least Common Multiples with Venn Diagrams

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Abstract: In this article the authors discuss the connections between the Greatest Common Factor and Least Common Multiple for two and three integers using Venn diagrams. The calculations of the GCF and LCM are related to the formulas for calculating the number of elements in the union of two and three sets.

The Number and Operations Standard for Grades 6-8 in NCTM's *Principles and Standards for School Mathematics* includes the expectation that students "use factors, multiples, prime factorization, and relatively prime numbers to solve problems." Many times, students are asked to find the greatest common factor (GCF) and the least common multiple (LCM) of two or more positive integers to demonstrate their understanding of factors, multiples, and prime factorization. For some students, this is simply a matter of memorizing a process and applying it correctly to each pair of positive integers without actually understanding the ideas behind the process. We want to present a visual approach to finding the GCF and the LCM of a set of positive integers.

Greatest Common Factors and Least Common Multiples

Common methods of finding the greatest common factor include:

1. Listing all of the factors of each integer, identifying the common factors, and choosing the largest of these common factors;
2. Finding the prime factorization of each integer, identifying the common prime factors, and multiplying the common prime factors to find the greatest common factor;

3. Once students have learned the meaning of exponents, using the prime power representation of each integer to identify common prime factors, choosing the smallest exponent of each common factor, and then multiplying common prime powers to obtain the greatest common factor; and

4. Using the Euclidean Algorithm (especially if the integers are large).

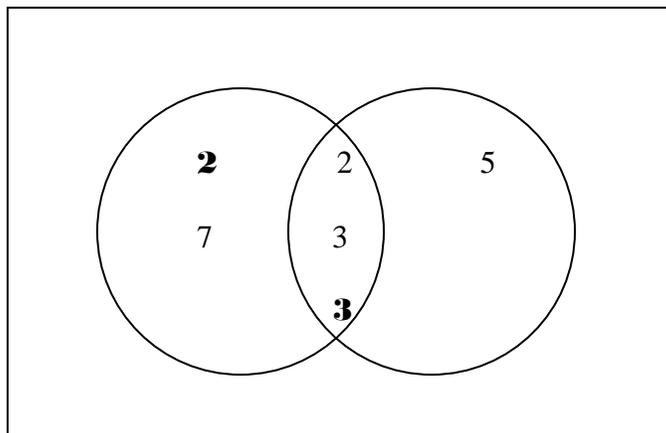
Each of these methods have advantages and disadvantages; one of the primary disadvantages of all of these methods is the lack of a visual aid that allows students to see why a particular process works. However, the second method, using the prime factorization of each integer, can be understood visually by using a Venn diagram. This method can be easily generalized to find the GCF and LCM of three or more integers.

We begin with an example involving a pair of integers. Suppose we want to find the GCF (252, 90). We find the prime factorization of each integer:

$$252 = 2 \cdot \mathbf{2} \cdot 3 \cdot \mathbf{3} \cdot 7$$

$$90 = 2 \cdot 3 \cdot \mathbf{3} \cdot 5.$$

Each successive occurrence of a prime factor is represented in a different font so we can account for repeated prime factors. We put the prime factors of 252 in set A and the prime factors of 90 in set B in a Venn diagram.



The product of the prime factors in set A is 252, the product of the prime factors in set B is 90, and the product of the prime factors in the intersection is the greatest common factor:

$$\text{GCF}(252, 90) = 2 \cdot 3 \cdot 3 = 18.$$

This idea also lends itself well to finding the least common multiple of two positive integers. The LCM of two integers is the product of all of the prime factors in the first number and the prime factors of the second number not included in the first number. Thus in our Venn diagram, the least common multiple is the product of all of the prime factors in the union of the two sets (all of the numbers that appear in the diagram):

$$\text{LCM}(252, 90) = 2 \cdot 7 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 1260.$$

The Relationship between GCF and LCM

A fundamental concept relating two positive integers, their greatest common factor, and their least common multiple is that the product of the two integers equals the product of the greatest common factor of these two integers and the least common multiple of these two integers. In symbols, if a and b are positive integers, then:

$$a \cdot b = \text{GCF}(a, b) \cdot \text{LCM}(a, b). \quad (1)$$

This relationship allows students to find the least common multiple of two positive integers by finding the product of the two integers and dividing by the greatest common factor of the two integers. Again, this allows students to memorize a process without understanding why the process works.

This concept has an interesting parallel with Venn diagrams. If A and B are sets with finite elements, then the number of elements in the union of A and B is equal to the sum of the number of elements in A and the number of elements in B minus the number of elements in

$A \cap B$; that is,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

This can be written as:

$$n(A) + n(B) = n(A \cap B) + n(A \cup B). \quad (2)$$

When we found the greatest common factor of 252 and 90, we multiplied the primes in the intersection of the two sets in the Venn diagram; when we found the least common multiple, we multiplied the primes in the union of the two sets in the Venn diagram. The product of the primes in set A is 252; the product of the primes in set B is 90.

$252 \cdot 90 = \text{GCF}(252, 90) \cdot \text{LCM}(252, 90)$ which is simply

(the product of the primes in A) \cdot (the product of the primes in B) = (the product of the primes in $A \cap B$) \cdot (the product of the primes in $A \cup B$).

This relationship parallels equation 2 with addition replaced by multiplication and the number of elements in the set replaced by the product of the elements in the set.

GCF and LCM of Three Positive Integers

A much more difficult problem for students is finding the greatest common factor and the least common multiple of three positive integers and understanding the relationship between these two concepts and the prime factorizations of the individual integers. Using Venn diagrams makes this process much easier.

We illustrate with an example. Suppose we want to find the greatest common factor of the integers 8085, 1575, and 990. We find the prime factorization of each number:

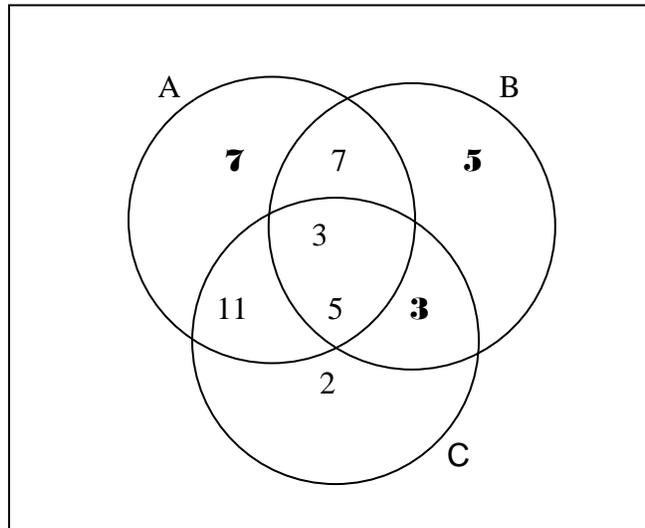
$$8085 = 3 \cdot 5 \cdot 7 \cdot \mathbf{7} \cdot 11,$$

$$1575 = 3 \cdot \mathbf{3} \cdot 5 \cdot \mathbf{5} \cdot 7,$$

and

$$990 = 2 \cdot 3 \cdot \mathbf{3} \cdot 5 \cdot 11.$$

We call the integers in the prime factorization of 8085 set A, the integers in the prime factorization of 1575 set B, and the integers in the prime factorization of 990 set C. Put these factors in a Venn diagram.



The greatest common factor of the three integers is the product of the primes in the intersection of all three sets. From the Venn diagram, it is easy to see that:

$$\text{GCF}(8085, 1575, 990) = 3 \cdot 5 = 15.$$

The least common multiple is the product of all the primes in the union of the three sets; that is,

$$\text{LCM}(8085, 1575, 990) = 7 \cdot \mathbf{7} \cdot \mathbf{5} \cdot 11 \cdot 3 \cdot 5 \cdot \mathbf{3} \cdot 2 = 242550.$$

Furthermore, this Venn diagram also allows us to see that:

$$\text{GCF}(8085, 1575) = 7 \cdot 3 \cdot 5 = 105,$$

$$\text{GCF}(8085, 990) = 3 \cdot \mathbf{3} \cdot 5 = 45,$$

and

$$\text{GCF}(1575, 990) = 11 \cdot 3 \cdot 5 = 165.$$

We now use this information and a process similar to the process described in the case of two positive integers to create a formula for the least common multiple of three positive integers. When counting the elements in three sets A, B, and C using Venn diagrams, we have the formula¹:

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) \\ &\quad - n(B \cap C) + n(A \cap B \cap C). \end{aligned}$$

This can be rewritten as:

$$\begin{aligned} n(A \cup B \cup C) + n(A \cap B) + n(A \cap C) + n(B \cap C) \\ = n(A) + n(B) + n(C) + n(A \cap B \cap C). \end{aligned} \quad (3)$$

We adopt the pattern that we used for two integers to relate the greatest common factor and the least common multiple of three positive integers. Unions are replaced by the least common multiple, while intersections are replaced by the corresponding greatest common factor. The additions in equation 3 are replaced with multiplication of the corresponding products of elements in the indicated sets. Thus the relationship for positive integers a, b, and c is:

$$\begin{aligned} \text{LCM}(a, b, c) \cdot \text{GCF}(a, b) \cdot \text{GCF}(a, c) \cdot \text{GCF}(b, c) \\ = a \cdot b \cdot c \cdot \text{GCF}(a, b, c). \end{aligned}$$

We may now solve for LCM(a, b, c):

$$\text{LCM}(a, b, c) = \frac{a \cdot b \cdot c \cdot \text{GCF}(a, b, c)}{\text{GCF}(a, b) \cdot \text{GCF}(a, c) \cdot \text{GCF}(b, c)}. \quad (4)$$

In our example, equation 4 becomes:

$$\text{LCM}(8085, 1575, 990) = \frac{8085 \cdot 1575 \cdot 990 \cdot 15}{105 \cdot 45 \cdot 165} = 242550,$$

¹ An interesting question is why this formula correctly counts the elements in the sets. See Anderson and Bell, *Number Theory with Applications*, pp. 301-302.

which agrees with our conclusion from the Venn diagram.

These same ideas generalize to find the least common multiple of any number of positive integers, simply by knowing how to find the number of elements in the union of sets by counting the elements in each set and the elements in the various intersections.

References

1. Anderson, James A. and Bell, James M. *Number Theory with Applications*. Prentice Hall, 1997.
2. National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA.: NCTM, 2000.

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