What is the Equals Sign?

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Algebra teachers often find students who treat the equals sign as a signal to write the answer. These students seem to think that the equals sign is like the horizontal line that we draw at the bottom of a column of figures when we wish to add them up. It says, "The answer comes next." Another dysfunctional thought-model connects the equals sign we write on paper with the equals-*key* on a calculator, which we press when we want the calculator to show us the answer.

In algebra, that's not how the equals sign functions. Many teachers address the problem by explaining that the equals sign must be understood like the verb "is" in a statement of fact, and not the verb "do" in a command. If students can get over the idea that "in math class, we write to find the answer" and realize that much of our writing aims to make complete, meaningful statements about numbers, figures and quantities, then the misunderstanding and misuse of the equals sign will cease, or so one hopes.

Teaching that the equals sign is used to make statements is only a partial remedy, based on only a partial diagnosis of the problem. It omits that very special feature of algebra that makes this use of the equals sign necessary. In arithmetic, there really is not much need to use the equals sign in its declarative mode. The command mode is adequate. After all, what's the difference between saying, "Add 3 and 5; you get 8," and saying, more symbolically,"3 + 5 = 8."

The problem is not that students arrive in algebra with a misconception about the equals sign. They have a conception that's quite appropriate in the world of arithmetic and which they have learned to use effectively in that context. But this conception needs to be modified to meet the demands of algebra. In algebra, it's not so important for students to *react* to statements like, "$5 = x$," and "$3 + x =$

8," but to *understand* how statements like these two are *related*. This means understanding the role that *variables* play.

To help students make the transition from arithmetic to algebra, teachers need a good grasp of what the whole "algebra game" is about. How do constant-symbols, variable-symbols and relation-symbols (like "=" and "≤") all work together in this game? What does this language-game do?

For many students, algebra is not a single game, but a lot of separate games: simplifying games, graphing games, solving games. In each of these kinds, there are many sub-kinds: solving linear equations, solving linear inequalities, solving pairs of linear equations, solving quadratics, *etc.* What is the meaningful core that holds all these things together? Where can we go to find a single story that tells us at once how all these things relate to one another, and why they demand a particular role for the equals sign and for variables? And what, precisely, is that role?

The creators of mathematical logic were deeply interested in how mathematical language works, and there is some evidence, I believe, that some of the logicians of the 19^{th} century were motivated by pedagogical concerns. The German mathematics pedagogue, Martin Ohm (brother of physicist Georg Ohm, remembered today for Ohm's law in electricity) wrote one of the earliest books on the development of number systems, foreshadowing Richard Dedekind's work, which cast the real number system in the form we know it today. Some historians have argued that foundational studies flourished in Europe the 19^{th} century in part because of the new demographics of the schools and universities demanded a better, clearer understanding of the foundational concepts in order to communicate to the rapidly expanding and ever-more diverse audience.

Pick up a good book on logic and read at least enough to understand what mathematical sentences are, the role that the equals sign plays in them, how the truth of a sentence is decided, and why statements with variables in them are sometimes not sentences *per se*, but "sentential functions."

Agreeing about the exact meaning of such things with the level of precision used in logic would be a very useful piece of knowledge for all of us to share. I can recommend Tarski's *Introduction to Logic and the Methodology of the Deductive Sciences* (1941), which treats all of this basic information in the first few pages. I hope that some readers will be "hooked" by the simplicity and clarity that Tarski offers, and will delve more deeply into this fine book.

I don't propose that logic should be taught to children, and I don't believe that logic should control everything about pedagogy. But logic does provide a very precise model of how mathematical language works (or should work—or *might* work), and a lot of actual mathematical discourse really is in a form of colloquial first-order logic. What I am saying is that logic provides a "foil" for school math. By foil, I mean (as Wikipedia, says) "a character that contrasts with another character, usually the protagonist, and so highlights various facets of the main character's personality." The protagonist, here, of course, is Algebra I.

James J. Madden, *professor of mathematics at Louisiana State University, Baton Rouge, has research interests in the areas of algebraic geometry and ordered algebraic structures. Professor Madden is also interested in supporting K12 mathematics teachers. He helped to found the [MathVision Lab](http://www.math.lsu.edu/~madden/mathvision/index.html) (http://www.math.lsu.edu/~madden/mathvision/index.html). He has also posted some resources for teachers of teachers at the [Secondary Math Site](http://www.math.lsu.edu/~madden/Sec_Math_Site/) (http://www.math.lsu.edu/~madden/Sec_Math_Site/).*