Pentominoes

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Abstract. *Pentominoes and their relatives the polyominoes, polycubes, and polyhypercubes will be used to explore and apply various important mathematical concepts. In this paper, the definition of the pentominoes will be examined. It will be proved that there are exactly 12 pentominoes. The pentominoes will then be represented as matrices with the manipulations of the pentominoes represented as matrix addition and multiplication. The concept of a pentomino's mod numbers will be introduced and, in an application of modular arithmetic, will be used to prove the existence and uniqueness of pentominoes and figures made with pentominoes as well as similar facts about polyominoes, polycubes, and polyhypercubes.*

The pentominoes are a simple-looking set of objects through which some powerful mathematical ideas can be introduced, investigated, and applied. The pentominoes can be expressed in an analytic geometric setting where they are changed into vectors and matrices with integral entries and manipulated as numbers. The questions about pentominoes can be transformed into equivalent questions in a different mathematical setting from which some interesting properties of the pentominoes can be proved. This is similar to the ideas in algebraic topology where questions about spheres and tori can be answered by looking at the algebraic structure of paths on their surfaces and converting the topological questions into algebraic ones. With pentominoes and their relatives, the objects are transformed into matrices and vectors to which linear algebra and number theory are applied.

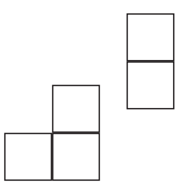
The pentominoes are a puzzle that has been used by teachers to introduce students to important math concepts such as symmetry, area, and perimeter. Pentominoes are suggested for use by teachers on page 99 of the NCTM Principles and Standards, in the Geometry Standard of the Pre-K-2 section. They appear as well in various NCTM articles, such as The Pentomino Square Problem in Mathematics Teaching in the Middle School, March 2003 p. 355 and the

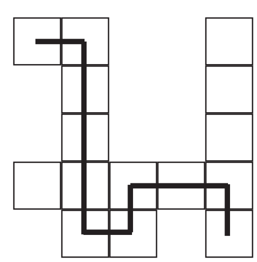
NCTM Illuminations Exploring Cubes Activity Sheet. In these articles, the following questions are investigated:

1. How many pentominoes are there?
2. Can a certain figure be made with pentominoes?
3. Can the pentomino be folded to make an open box?

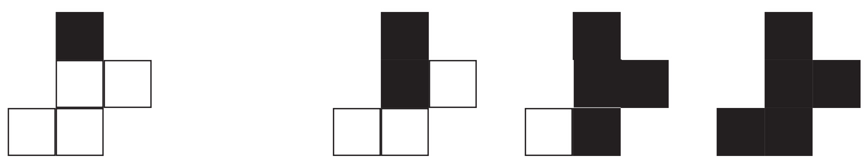
This paper will look at the first two questions but will go into them in much greater depth using more advanced math, namely linear algebra and modular arithmetic, to prove some surprising results.

First there must be a definition of pentomino. The above activities give muddled or incorrect definitions. The usual definition, such as the one given in the NCTM's Principles and Standards is incorrect and yields an infinite number of pentominoes. On p.99, it is stated, "For example, a second-grade teacher might instruct the class to find all the different ways to put five squares together so that one edge of each square coincides with an edge of at least one other square (see fig. 4.15)." By that definition, the figure at the left is a pentomino because one edge of each square coincides with an edge of at least one other square. Definitions must be made carefully so that all the squares in the figure are connected. A better definition would be that a polyomino is a plane figure made of squares such that two different squares can touch only on sides which coincide and for every two squares in the polyomino, there is a path that goes through adjacent squares from the first square chosen to the last. The figure at the right shows a path from one square to another in a large polyomino.



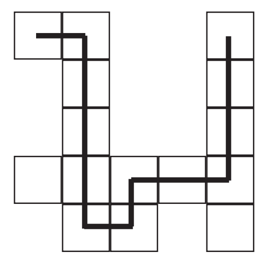


Students could be asked for different tests that would show this connectedness. Do they have to find a path between any two squares in a figure to show that it is a polyomino or can they find a simpler test that requires fewer paths? The number of such paths between any two squares in a polyomino is the “handshake problem” and requires n(n-1)/2 checks to see if it is true for a polyomino with n squares. If one square is connected by a path to every other square, only n-1 checks are required. This is sufficient because if there is a path from square a to square b and a path from square b to square c, then there is a path from a to c going through b. Students could be asked to find other tests. For example, color a square in a polyomino and color all the squares in the polyomino that touch the colored square. Color all the squares that touch a colored square. Continue coloring squares in this manner in the polyomino as shown below. If all the squares in

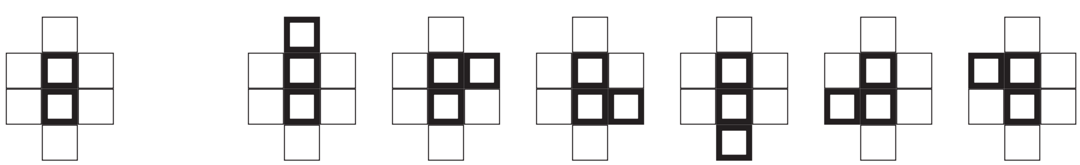


the figure are eventually colored then it is a polyomino. This test would require at most n-1 steps in coloring. Students could be asked for a proof of this assertion.

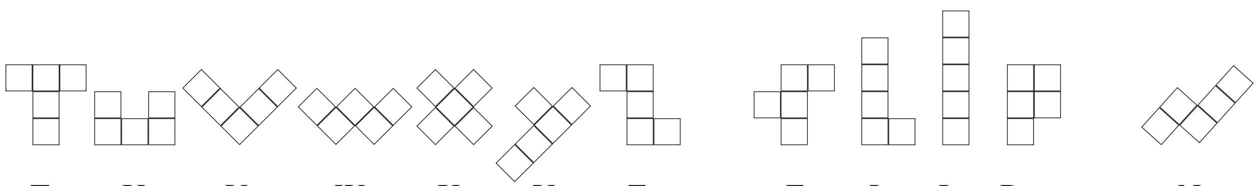
The question, “Are there exactly 12 pentominoes?” can now be answered. All the pentominoes can be found by looking at the tetraminoes, that is, polyominoes with 4 squares, adding one square to each of them, and throwing out the duplicates. In general, the n+1-ominoes can be found by looking at the n-ominoes, adding one square to each of them in every possible way, and throwing out the duplicates. How is this shown? Take an n+1-omino. If there is a square in that n+1-omino that can be removed and the remaining n squares are still connected, that is, is an n-omino, then it has been shown that the n+1-omino can be made by adding one square to an n-omino. Take all the paths from one square in the n+1-omino to another square in the n+1-omino which do not go through the same square twice. Since there are a finite number of squares in a polyomino, there is a longest such path. The end squares can be removed with the remaining squares of the original n+1-omino still being connected, that is the remaining squares form an n-omino for the following reason. If the remaining squares did break into two disconnected pieces, then the longest path would be in one of the pieces. But there would be a longer path in the original n+1-omino by connecting the end point that we removed to a square in the disconnected piece not containing the path. This is a contradiction, because the path originally choosen was the longest path in the n+1-omino and the constructed path is longer. Thus, the n+1-omino can be formed from an n-omino by adding one square. The path in the polyomino at the right is the longest possible and both its endpoints are removable.



There are now a specific number of pentominoes to check by adding one square to all of the tetraminoes or 4-ominoes and throwing out duplicates. This process is iterative, there is only one monomino or 1-omino. Generate the one domino or 2-omino from the 1-omino; the two 3-ominoes from the 2-omino; the five 4-ominoes from the two 3-ominoes; and finally the twelve pentominoes from the five 4-ominoes. The problem comes down to eliminating the duplicates. Here are all the ways to add a square to the domino or 2-omino.

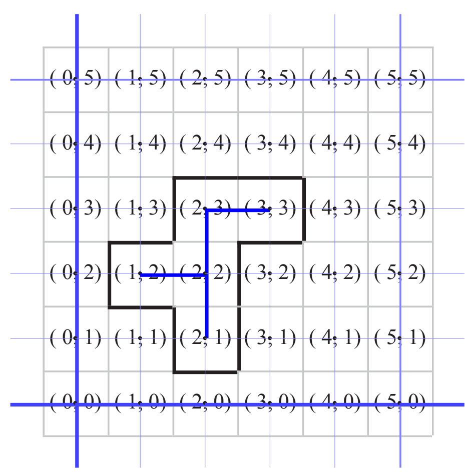


Students could show all these and then decide which are the same. Have the students explain why they think that two of the figures are the same by having them show how they would move one figure to the other. This can lead into a discussion of rigid motions in the plane, that is, rotation and reflection as well as translation. The students can continue with this process to obtain the 12 pentominoes. Some students might want to continue and find the 35 hexominoes. Here are the 12 pentominoes and their common letter names:



T U V W X Y Z F L I P N

If linear algebra and number theory are to be used with the pentominoes, then somehow numbers must be attached to each pentomino. Let the sides of the squares in the pentominoes be length 1. Look at a coordinate plane and take the grid of all points with integral coordinates. If you place a pentomino on the plane so that the center of each of its squares is on a grid point, then the pentomino piece could be written as a matrix consisting of 5 ordered pairs. Consider the piece in the figure to the right. This is the F pentomino. You can consider the piece in a number of ways. First, it is the five squares outlined by the heavy black line. Second, it is the figure represented by the heavy blue lines.



$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 2 \\ 2 & 3 \\ 3 & 3 \end{bmatrix}$$

Third, it is the matrix shown on the left. If we move the piece, we would change the matrix representing the piece, but we would change the matrix in a way that was representable as matrix operations.

$\begin{bmatrix} a & b \\ a & b \\ a & b \\ a & b \\ a & b \end{bmatrix}$ For example, adding the matrix on the left to a pentomino matrix would move the pentomino right a squares and up b squares.

Multiplying a pentomino matrix on the right by these matrices does the following:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Do not move the pentomino.

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Rotate clockwise 90^0.

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ Rotate clockwise 180^0 or Reflect through the origin.

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ Rotate clockwise 270^0 or Rotate counter-clockwise 90^0.

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ Reflect through the y-axis.

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Reflect through the x-axis.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Reflect through the line y = x.

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ Reflect through the line y = -x.

It is the objective to find some number or numbers that distinguish each pentomino. It is not at all obvious that the following matrices all represent the same pentomino, the F pentomino:

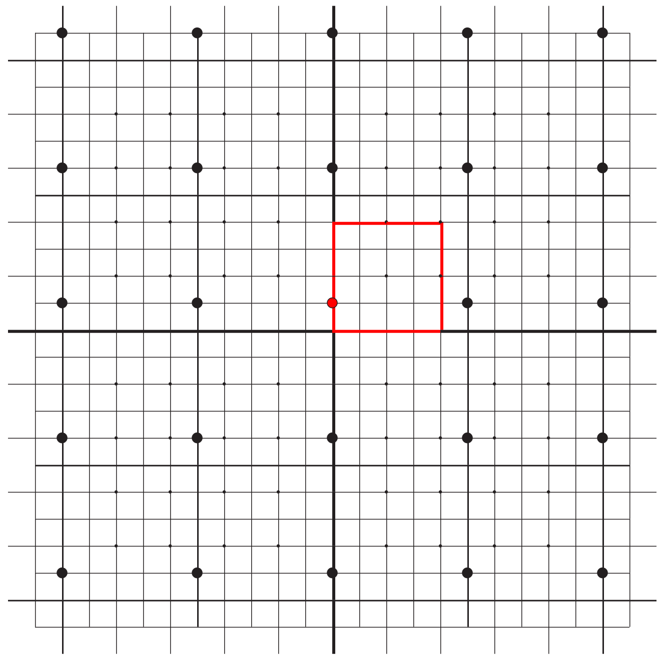
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 2 \\ 2 & 3 \\ 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 4 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 3 \\ 3 & 3 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & -2 \\ -3 & -2 \\ -3 & -4 \\ -3 & -3 \\ -4 & -3 \end{bmatrix}$$

Even shuffling the rows makes it difficult to see that the first and third matrices contain the same points. It is important to make it more obvious. Adding the columns to get a single vector will give the same result even if the rows are shuffled.

How will this affect the other matrices representing the F pentomino? If the pentomino is translated over a to the right and b up we have the following:

$$\underset{(10,\ 11)}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 2 \\ 2 & 3 \\ 3 & 3 \end{bmatrix}} + \begin{bmatrix} a & b \\ a & b \\ a & b \\ a & b \\ a & b \end{bmatrix} = \underset{(10+5a,\ 11+5b)}{\begin{bmatrix} 2+a & 1+b \\ 1+a & 2+b \\ 2+a & 2+b \\ 2+a & 3+b \\ 3+a & 3+b \end{bmatrix}}$$

Translating the F pentomino over and up gives a a vector that differs from the original vector by a multiple of 5 in both coordinates. If the vector mod 5 is taken, then the vector of any translation mod 5 stays the same. This is called a mod number for the particular pentomino. For the example of the F pentamino above with vector (10,11), the mod number is (0,1) since (10,11) = (5*2+0, 5*2+1). Any translate of this particular placing of the F pentomino would have a mod number of (0,1). The black dots on the grid to the right show all the vector sums of translates that this particular orientation of the F pentomino could be. The small red square in the first quadrant of the grid shows all the possible locations for mod numbers of any pentomino. The red dot is a mod number for this particular orientation of the F pentomino.



How do reflections and rotations affect the mod number? Since multiplying a vector by a matrix is linear, the mod number is multiplied by the matrix. Since we are only dealing with interchanging columns and multiplying by 1 or -1, the mod number for the new position of the F pentomino will be the mod number of the original position multiplied by the matrix corresponding to the reflection or rotation.

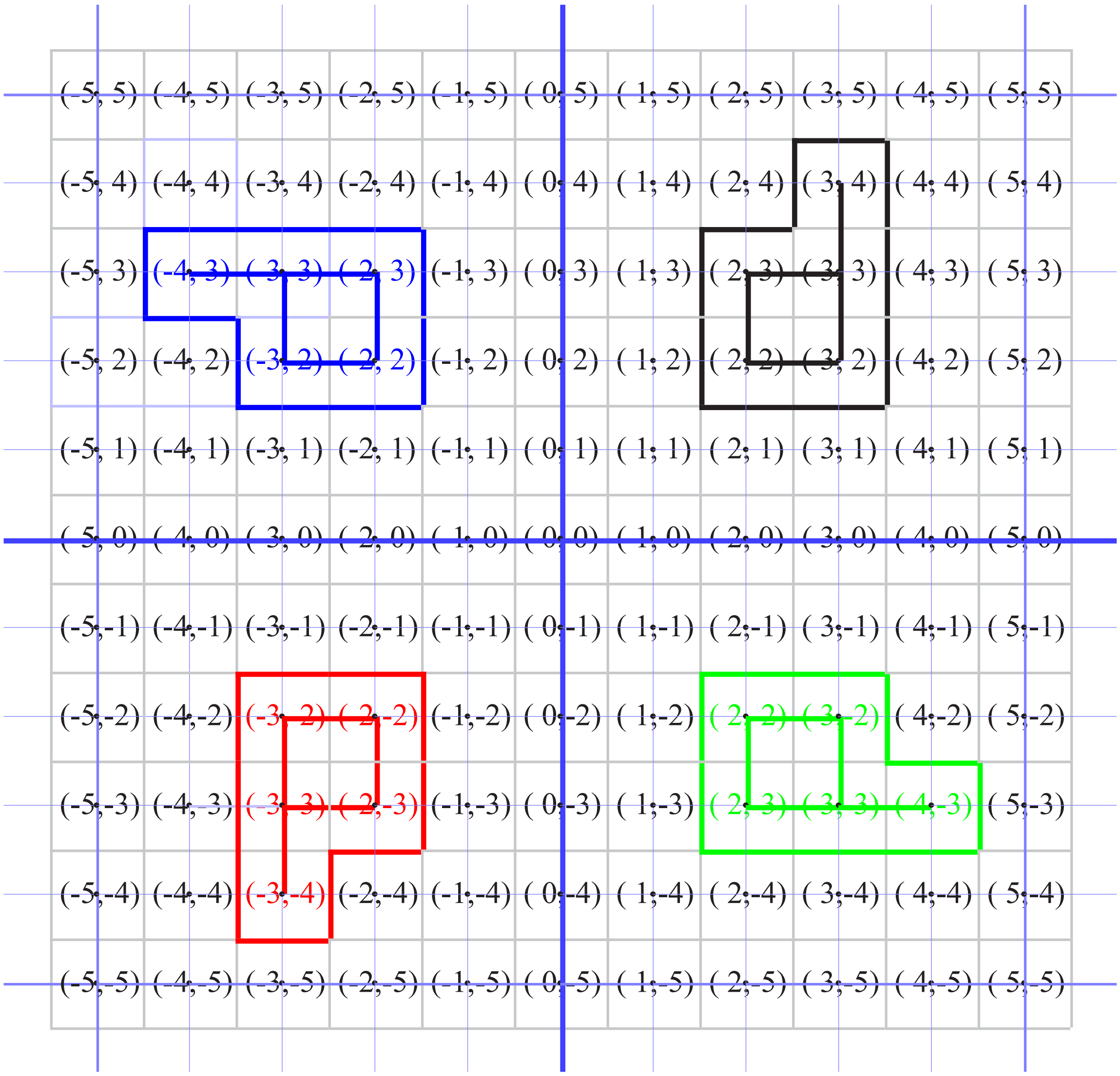
Why are the mod numbers of any use in working with the pentominos. The only possible mod numbers that a piece can have are obtained by taking the mod numbers for any orientation of the piece and then multiplying those mod numbers by the matrices for the eight reflections and rotations. Two pentominoes can be the same pentomino only if they have the same mod numbers. This is not crucial with the pentominoes because visual inspection can quickly determine if two pentominoes are the same except for rotations and reflections. However when dealing with something like polyhypercubes, say with 10 cubes in 6 dimensions, then the mod numbers will be vectors with 6 coordinates mod 10. There might be some overlap of mod numbers between pieces, but the mod numbers considerably reduce the number of other polyhypercubes needed to be checked for duplication as well as indicating a definite procedure to follow for the check, that is, orient so that the mod numbers are equal, dictionary order, and check for equality. A computer could be used to go rapidly through the possibilities.

A second use which has many more applications with the pentominoes would be showing whether a shape is impossible to make with the pentominoes. Since any shape formed with pentominoes consists of a multiple of 5 squares, the mod number for this shape is defined and is equal to the sum of the mod numbers of the pieces. Take the 6 by 10 rectangle, which can be formed with the 12 pentominoes. Let the pentominoes be placed on the grid. If the sum of the mod numbers of the pentominoes does not equal (0,0), the mod number of the rectangle, the pentominoes in those orientations can not form the rectangle by translation without rotation or reflection. It is difficult to stop from physically rotating or turning over a piece when working with the pentominoes, but the mod numbers would be very useful when solving for a shape using a computer program.

On the remaining pages, the mod numbers for the P pentomino are shown. One position of the P pentomino is shown in the first quadrant along with the seven other mod numbers that are obtained by reflections and rotations. The mod number classes for the different pentominoes are then shown on the following page. The paper finishes with several examples using mod numbers to construct shapes with pentominoes and prove the impossibility of certain constructions with pentominoes and dominoes.

$$\begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 3 \\ -4 & 3 \\ -3 & 3 \\ -3 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix}$$

$$(3 \quad 4) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = (1 \quad 3) \qquad (3 \quad 4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (3 \quad 4)$$



$$\begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -3 & -2 \\ -3 & -4 \\ -3 & -3 \\ -2 & -3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -3 \\ 4 & -3 \\ 3 & -3 \\ 3 & -2 \end{bmatrix}$$

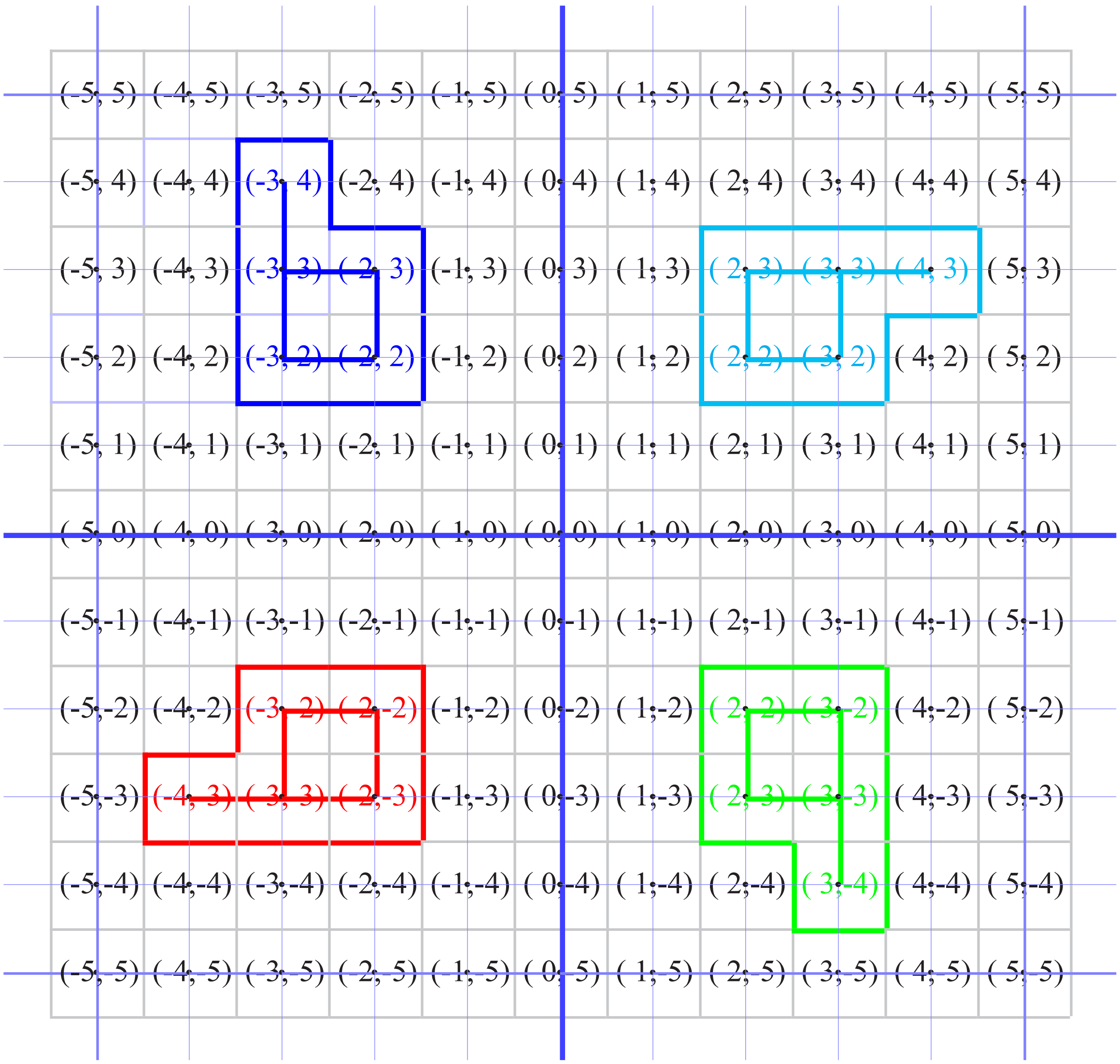
$$(3 \quad 4) \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = (2 \quad 1) \qquad (3 \quad 4) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = (4 \quad 2)$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -3 & 2 \\ -3 & 4 \\ -3 & 3 \\ -2 & 3 \end{bmatrix}$$

$$(3 \quad 4) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = (2 \quad 4)$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \\ 4 & 3 \\ 3 & 3 \\ 3 & 2 \end{bmatrix}$$

$$(3 \quad 4) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (4 \quad 3)$$

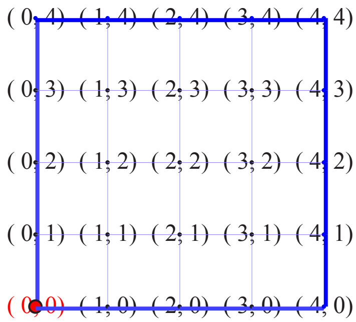


$$\begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & -3 \\ -4 & -3 \\ -3 & -3 \\ -3 & -2 \end{bmatrix}$$

$$(3 \quad 4) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = (1 \quad 2)$$

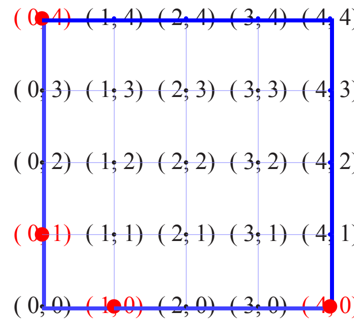
$$\begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 3 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & -2 \\ 3 & -4 \\ 3 & -3 \\ 2 & -3 \end{bmatrix}$$

$$(3 \quad 4) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = (3 \quad 1)$$

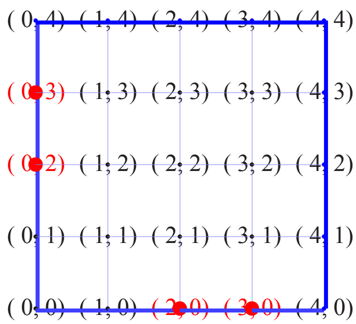


The pentominoes separate into 6 classes of pieces.

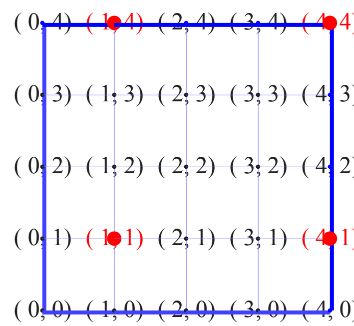
The first class is the class containing I, X, and Z. This class has the mod numbers (0 0).



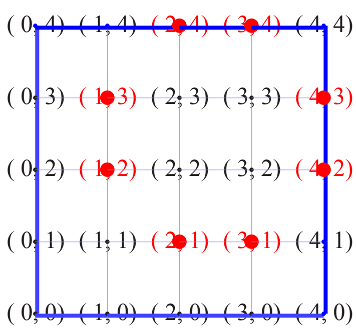
The second is the class containing F. This class has the mod numbers, (0 1), (0 4), (1 0), and (4 0).



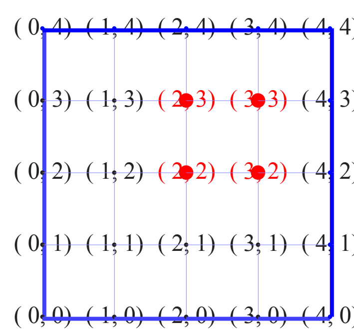
The third is the class containing T and U. This class has the mod numbers, (0 2), (0 3), (2 0), and (3 0).



The fourth is the class containing W and L. This class has the mod numbers, (1 1), (1 4), (4 1), and (1 4).



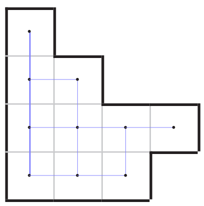
The fifth is the class containing Y and P. This class has the mod numbers, (1 2), (1 3), (2 1), (2 4), (3 1), (3 4), (4 2), and (4 3).



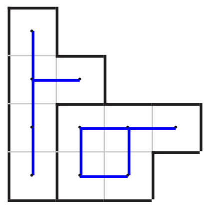
The sixth is the class containing V and N. This class has the mod numbers, (2 2), (2 3), (3 2), and (3 3).

The mod numbers can be used to see whether certain constuctions with the pentominoes are impossible. It all depends on the fact that the sum of the mod numbers is the mod number of the sum, that is the figure formed by the two pentominoes. Have students discuss why this is so.

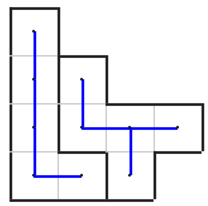
Here are some examples.



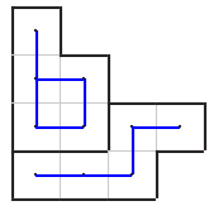
(0,1)



(1 3)+(4 3)=(0 1)
Y + P

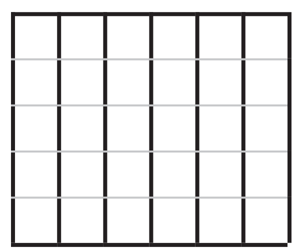


(1 1)+(4 0)=(0 1)
L + F



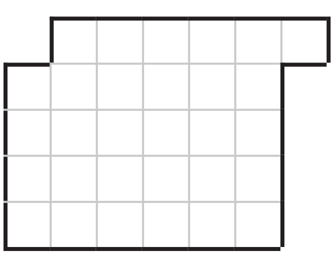
(2 4)+(3 2)=(0 1)
P + N

The mod numbers can be used to show that it is impossible to solve a puzzle. Consider this example using mod numbers to show that it is impossible to solve certain pentomino problems. The figure at the right shows that the figure, a 5 x 6 rectangle is constructed of 6 I s. Can the figure formed by moving one of the squares as shown below be constructed if any number of I s, X s, and Z s as well as at most one other piece other than an F is used?.



(0 0)

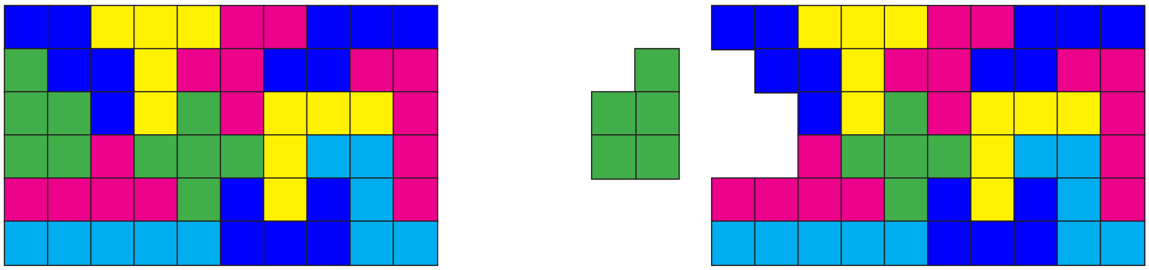
Oddly enough, there is no solution to this puzzle. I, X, and Z all have mod numbers (0 0).



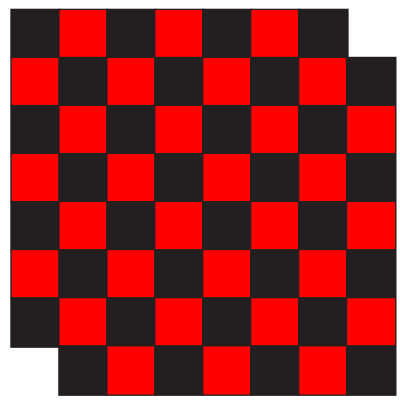
(1 0)

The sum of the mod numbers for the final shape must be the mod numbers of the other piece. The mod number of the one other piece must be (1 0). Since F is the only piece with this mod number, the shape on the left cannot be constructed with one piece that is not an F and all the other five pieces being I s, X s, or F s.

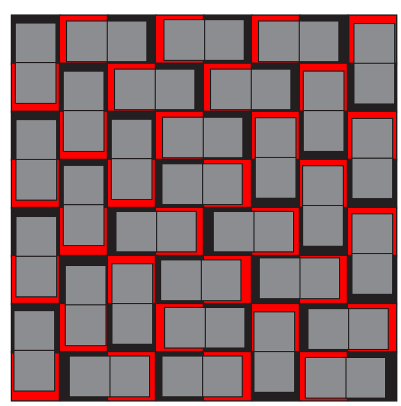
As an exercise, take this solution of the 6 x 10 using the twelve pentominoes. Flip the P pentomino as shown and prove the twelve pentominoes cannot be reassembled into the 6 x 10 through translations alone.



There is a well-known problem involving a checkerboard and dominoes. If each domino covers exactly two squares on a checkerboard and two diagonal corners are cut out, can it be covered using 31 dominoes? The answer is no because each domino covers a red and a black square on the checkerboard. The figure on the right has 30 red squares and 32 black squares, so 30 dominoes will always leave two black squares to be covered by domino 31. This will not work. Hence there is no covering. There is a similar problem using mod numbers.

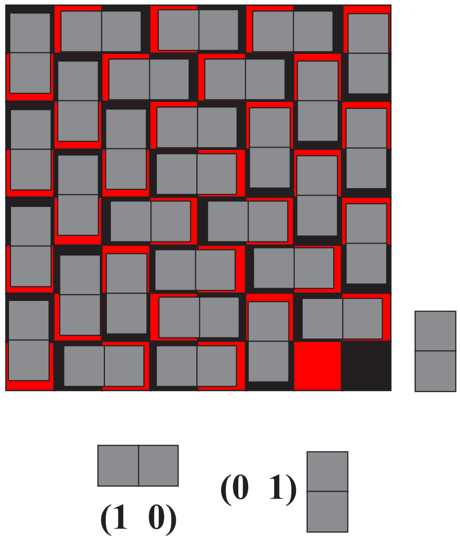


Cover a checkerboard with oriented dominoes such as shown on the right. There are two possible orientations, horizontal and vertical. Take one of the horizontal dominoes and replace it with a vertical one as shown below. Can the dominoes be put back without changing their orientations and still cover the checkerboard? The answer is no. Apply the same arguments



(0 0)

as before with the mod numbers but now use mod 2 instead of mod 5. The vertical domino has



mod number (0 1); the horizontal domino has mod number (1 0). The overall shape has a mod number of (0 0). If we take out a horizontal domino the remaining 31 dominos will form a shape that has a mod number (1 0), when we add the vertical domino we get a shape with a mod number of (1 1). The dominoes with exactly one with a changed orientation cannot be put back onto the checkerboard in their new orientations because the shape they make has mod numbers (1 1) and the mod numbers of the checkerboard are (0 0).

In this article, pentominoes and their close relatives have been explored. Pentominoes have been defined more carefully and the idea of paths through the figures has been discussed. Using the idea of a longest path, it has been shown that n+1-omines are derived from n-ominoes. This was used to show how to prove that there are exactly 12 pentominoes. It was shown how to assign coordinates to a polyomino and develop mod numbers for each pentomino. Properties of mod numbers were developed and used to prove several constructions would be impossible. An example was used to show how to develop and use mod numbers for dominoes and it was indicated how mod numbers could be extended to more dimensions and numbers of squares, cubes, hypercubes....

These investigations could also be used to introduce students to different important techniques and branches of mathematics. Proof by contradiction, iteration, and math induction were some of the techniques employed. Coordinates from analytical geometry, linear algebra, modular arithmetic from number theory, and group theory figure prominently in the arguments. For example, the mod numbers could be used to explore symmetry from the point of view of linear algebra. The pentominoes also open an avenue into group theory to explore the operation of a group, the eight matrices for rotation and reflection, on a set, the mod numbers, to look at the various orbits of the mod numbers and the breaking down of the mod numbers into equivalence classes. Even problems in computer science could be investigated. If the pentominoes were to be assembled into various constructions, the mod numbers could be used to eliminate many possible cases to be tried and improve the efficiency of algorithms looking for solutions. The use of the matrices could extend the problems investigated with more efficient computer programs from the pentominoes to problems with polyominoes as well as higher dimension cubes and hypercubes.

There is more to pentominoes than area, perimeter, and simple symmetry and the techniques used to study them can extend well beyond what is taught in middle school.

Bio - Bruce Baguley works for Cascade Math Systems, LLC. He received a BA in Mathematics from Tulane University, an MS in Mathematics from MIT, and his teacher training from Heritage College in Toppenish, WA. While teaching elementary and middle school students, he became interested in showing math concepts using manipulatives rather than relying on memorizing formulas. He has given numerous workshops at math conferences over the past few years, showing people how to use manipulatives to represent math concepts from counting, through whole number, rational, and integer operations, to solving and graphing linear equations as well as proving number theory problems.

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