DISCRETE MATHEMATICS FOR ALL

Overview of Discrete Mathematics in Prekindergarten through Grade 12

Guest Editorial

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Adapted from NCTM's forthcoming *Navigations* books on discrete mathematics:
Navigating Through Discrete Mathematics in Prekindergarten to Grade 5 and
Navigating Through Discrete Mathematics in Grades 6 to 12
(DeBellis, Rosenstein, Hart, and Kenney 2009;
Hart, Kenney, DeBellis, and Rosenstein 2008)

The world we live in is changing rapidly; mathematical knowledge and applications are growing dramatically. Yet the school mathematics curriculum changes only slowly and reluctantly. To keep up with the rapidly changing, high-technology world that students live in, the mathematics curriculum must include both *legacy content* and *future content* (cf. Prensky 2001). *Legacy content* is the important content that has withstood the test of time and deserves to remain in the curriculum, such as important topics in arithmetic, algebra, and geometry. *Future content* is the content that students need now and in the future to be prepared for the modern world. Such future content includes important topics in discrete mathematics.

This article provides an overview of discrete mathematics for prekindergarten through grade 12. The article is adapted directly from the Introduction in the forthcoming NCTM *Navigations* books on discrete mathematics, the first of which, *Navigating Through Discrete Mathematics in Grades 6 to 12* (Hart, Kenney, DeBellis, and Rosenstein 2008), will be released at the annual NCTM conference in April 2008.

Discrete mathematics is an important branch of contemporary mathematics that is widely used in business and industry. Elements of discrete mathematics have been around as long as mathematics itself. However, discrete mathematics only emerged as a distinct branch of mathematics in the mid-1900's, motivated most strongly by the computer revolution, but also by the need for mathematical techniques to help plan and implement such huge logistical projects as World War II and landing a man on the moon. It has grown to be even more important and pervasive today.

NCTM's *Principles and Standards for School Mathematics* recommends that **"discrete mathematics should be an integral part of the school mathematics curriculum"** (NCTM, 2000, p. 31). There are two major changes in the recommendations for discrete mathematics from NCTM's 1989 *Curriculum and Evaluation Standards for School Mathematics*, which include a Discrete Mathematics Standard for grades 9-12, and the 2000 *Principles and Standards*. First, discrete mathematics is now recommended for all grades, from prekindergarten through grade 12. Second, there is no longer a separate standard for discrete mathematics. Rather, the main topics of discrete mathematics are distributed across the Standards, and thus interwoven throughout the other strands of mathematics.

In this article, we present an overview of discrete mathematics and suggest guidelines for integrating discrete mathematics topics into an NCTM-Standards-based curriculum. Since discrete mathematics may be unfamiliar to many readers, we begin by considering the question of just what is discrete mathematics.

What Is Discrete Mathematics?

Discrete mathematics is often described by listing the topics it includes, such as vertex-edge graphs, systematic counting, iteration and recursion, matrices, voting methods, and fair division. College discrete mathematics courses may contain these topics and others like sets, relations, difference equations, and Boolean algebra. In general, discrete mathematics is concerned with finite processes and discrete phenomena. It is sometimes described as the mathematical foundation of computer science, but in fact it has even broader application than that, since it is also used in the social, management, and natural sciences. Discrete mathematics can be contrasted with continuous mathematics, such as the mathematics underlying most of calculus. But this association gives the impression that discrete mathematics is only for advanced high school students, while in reality elements of discrete mathematics are accessible and important for all students in all grades.

A broad definition of discrete mathematics might be that it is “the mathematics of finite decision making” (NCTM, 1990), or that it is the mathematics used to optimize finite systems. Common themes in discrete mathematics are *discrete mathematical modeling* – using discrete mathematical tools such as vertex-edge graphs and recursion to represent and solve problems, *algorithmic problem solving* – designing, using, and analyzing step-by-step procedures to solve problems, and *optimization* – finding the best solution. (For further discussion of discrete mathematical modeling, see Rosenstein 2006; for more on algorithmic problem solving, see Hart 1998; and for more about the question, What is Discrete Mathematics?, see Maurer 1997, and Rosenstein 1997.)

Which Discrete Mathematics Topics are Included in *Principles and Standards*?

Three key topics of discrete mathematics are integrated within NCTM’s *Principles and Standards*: combinatorics, iteration and recursion, and vertex-edge graphs.

- Combinatorics is the mathematics of systematic listing and counting.

- Combinatorics is used to solve problems such as determining the number of different orders in which three friends can be picked up, or counting the number of different computer passwords that can be created with five letters and two numbers.
- Iteration and recursion can be used to represent and solve problems related to sequential step-by-step change, such as the growth of population or money from year to year. To iterate means to repeat, thus iteration involves repeating a procedure, process, or rule over and over. Recursion is the method of describing the current step of a process in terms of previous steps.
- Vertex-edge graphs are geometric models consisting of points (called vertices) and line segments or arcs (called edges) that connect some of the points. Such graphs are used to model and solve problems about paths, networks, and relationships among a finite number of elements.

The focus of *Principles and Standards* is to integrate discrete mathematics into other areas of the mathematics curriculum. For example, vertex-edge graphs are an important part of geometry, recursion is often used in algebra, and systematic listing and counting are important in probability.

Matrices are often considered to be part of discrete mathematics, and matrices appear throughout *Principles and Standards*. Other discrete mathematics topics that may be included in the school curriculum include the mathematics of information processing (e.g., error-correcting codes and cryptography), and the mathematics of democratic and social decision making (e.g., voting methods, apportionment, fair division, and game theory). In this article, we will focus on the three discrete mathematics topics emphasized in *Principles and Standards*: combinatorics, iteration and recursion, and vertex-edge graphs (NCTM, 2000, p. 31). Before getting started on that, however, let's consider why discrete mathematics should be included in the school curriculum.

Why Should Discrete Mathematics Be Included in the School Curriculum?

Instructional time is valuable and there is limited space in the mathematics curriculum, so careful choices must be made about what to include. Discrete mathematics should be an integral part of the school mathematics curriculum because, in brief, it is useful, contemporary, and pedagogically powerful.

Discrete mathematics is useful mathematics. It is widely used in business, industry, and daily life. Discrete mathematics topics are "used by decision-makers in business and government; by workers in such fields as telecommunications and computing that depend upon information transmission; and by those in many rapidly changing professions involving health care, biology, chemistry, automated manufacturing, transportation, etc. Increasingly, discrete mathematics is the language of a large body of science and underlies decisions that individuals will have to make in their own lives, in their professions, and as citizens" (Rosenstein, Franzblau, and Roberts, 1997).

Discrete mathematics is contemporary mathematics. Discrete mathematics is an important and growing field of mathematics. It is particularly relevant in today's digital information age. For example, it underlies many aspects of the Internet, from securely encoding your credit card number when you make a purchase online to effectively compressing and decompressing the music, photos, and videos you download. Moreover, there are solved and unsolved problems at the frontiers of discrete mathematics that are not only relevant to today's students, but are accessible to them in that they can understand the problem and some partial solutions, such as the problem of finding the shortest circuit through a network (the Traveling Salesman Problem), or finding a more secure method for transmitting data between computers. Furthermore, since discrete mathematics is strongly linked to technology and today's school children are tomorrow's technological work force, it is important for their futures, as well as the future of our nation, that they become more familiar with the topics of discrete mathematics.

Discrete mathematics is pedagogically powerful. Discrete mathematics is not only important mathematical content, but it is also a powerful vehicle for teaching and learning mathematical processes and for engaging students in doing mathematics. Because discrete mathematics is useful and contemporary, it is often motivating and interesting for students. Discrete mathematics topics can engage and provide success for students who previously may have been unsuccessful or alienated from mathematics. Many of these topics are accessible to students in all grades, whether they are sorting different types of buttons in the early grades, or counting different flag patterns in middle school, or planning a school dance in high school using vertex-edge graphs and the critical path technique.

Furthermore, discrete mathematics is an effective context within which to address NCTM's Process Standards. Students strengthen their skills in reasoning, proof, problem solving, communication, connections, and representation in many ways. For example, they reason about paths in the visual context of vertex-edge graphs and justify why certain circuits must exist or not. They argue about why a recursive formula is better than an explicit formula, or vice versa, in a particular situation. They learn new methods of proof, such as proof by mathematical induction. They develop new types of reasoning, such as combinatorial reasoning, which is used to reason about how many different possibilities can arise in counting situations like the number of different pizzas that are possible when you choose two out of five toppings. Students exercise their problem solving skills as they solve problems in a variety of accessible yet challenging settings. They develop new problem-solving strategies, such as algorithmic problem solving (devising, using, and analyzing algorithms to solve problems), and new ways of thinking, such as recursive thinking. Students acquire and apply new tools for representing problems, like recursive formulas and vertex-edge graph models. Thus, students learn important mathematical content and powerful mathematical processes as they study discrete mathematics.

Recent History and Resources

Discrete mathematics surfaced as a curriculum issue in the 1980's when the Mathematical Association of America began debating the need for more discrete mathematics in the first two years of college, culminating in a report released in 1986 (MAA, 1986). Although the full recommendations of this report have not been achieved, more discrete mathematics courses have been instituted and continue to be taught in colleges around the world. In particular, Discrete Mathematics is now a standard course in collegiate computer science programs and is required for many mathematics majors.

This college-level discussion about discrete mathematics reached the high school level a few years later when the National Council of Teachers of Mathematics recommended a Discrete Mathematics Standard for grades 9-12 in its seminal document, *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989).

Stimulated by these Standards, discrete mathematics expanded rapidly in the school curriculum. The National Science Foundation funded teacher enhancement projects to help implement the discrete mathematics standard (for example, Hart 1987-1994, Kenney 1992-97, Rosenstein and DeBellis 1990-2005, and Sandefur 1992-95). High schools began offering courses in discrete mathematics, and many states added discrete mathematics to their state frameworks. High school discrete mathematics courses play an increasingly important role in the curriculum, providing essential mathematics for the technology- and information-intensive 21st century, particularly as more students are required to take more mathematics and yet not all students are best served by the traditional calculus-prep high school curriculum.

Several NSF-funded Standards-based curriculum development projects have integrated discrete mathematics into new high school textbooks (for example, *Core-Plus Mathematics* (Hirsch, et al. 1996-2008), and *Mathematics Modeling Our World* (COMAP 1998–2000)). In addition to the large and growing number of college discrete mathematics textbooks, several new high-school-appropriate textbooks emphasizing discrete mathematics have been published (for example, COMAP 2006, Crisler and Froelich 2005, and Tannenbaum 2007). At the elementary and middle school levels, there are books for teachers on discrete mathematics (for example, DeBellis and Rosenstein 2008). Finally, there are many articles about and activities for teaching discrete mathematics in journals, books, and on the Web.

So far we have considered what discrete mathematics is, along with some history and resources, and why discrete mathematics should be part of the curriculum. We will spend the rest of this article presenting an overview across the grades of the three main topics: combinatorics (systematic listing and counting), vertex-edge graphs, and iteration and recursion. It is important to note that in developing these topics across the grade levels, we have in mind two important progressions through the preK-12 curriculum – from concrete to abstract and from informal reasoning to more formal reasoning.

Overview of Systematic Listing and Counting in Prekindergarten through Grade 12

Students at all grade levels should be expected to solve counting problems. For example,

- In the early grades: How many different outfits can be put together that use one of three shirts and one of two pairs of shorts?
- In the middle grades: How many different four-block-high towers can be built using red and blue blocks?
- In the secondary grades: How many computer passwords are possible using six letters and three digits?

The key to answering such questions is to develop strategies for listing and counting, *in a systematic manner*, all the ways to complete the task. As students advance through the grade levels, the tasks will change – the objects to be counted will be abstract as well as concrete, the number of objects to be counted will increase, the representations will become more algebraic, and the reasoning will become more formal (culminating in proof) – but the common thread will be that the counting needs to be done systematically. If students have enough opportunity to explore counting problems at all grade levels, then these transitions will be smooth and understanding will be deeper. Further, understanding these counting strategies will help lay the necessary foundation for understanding ideas of probability.

Concepts and methods of systematic listing and counting are integrated within all the NCTM Standards. Consistent with this integration, the following recommendations suggest how systematic listing and counting can be developed throughout the grades.

Recommendations for Systematic Listing and Counting in Grades Pre-K–12

In prekindergarten through grade 2 all students should:

- Sort, organize, and count small numbers of objects
- Informally use the addition principle of counting
- List all possibilities in counting situations
- Sort, organize, and count objects using Venn diagrams

In grades 3–5 all students should:

- Represent, analyze, and solve a variety of counting problems using arrays, systematic lists, tree diagrams, and Venn diagrams
- Use and explain the addition principle of counting
- Informally use the multiplication principle of counting
- Understand and describe relationships among arrays, systematic lists, and tree diagrams, and the multiplication principle of counting

In grades 6–8 all students should:

- Represent, analyze, and solve counting problems that do or do not involve ordering and that do or do not involve repetitions
- Understand and apply the addition and multiplication principles of counting, and represent these principles algebraically, including with factorial notation
- Solve counting problems using Venn diagrams, and algebraically represent the relationships shown by a Venn diagram
- Construct and describe patterns in Pascal's triangle
- Implicitly use the pigeonhole principle and the inclusion/exclusion principle

In grades 9–12 all students should:

- Understand and apply permutations and combinations
- Use reasoning and formulas to solve counting problems in which repetition is or is not allowed and ordering does or does not matter
- Understand, apply, and describe relationships among the Binomial Theorem, Pascal's triangle, and combinations
- Apply counting methods to probabilistic situations involving equally-likely outcomes
- Use combinatorial reasoning, including for proofs

Overview of Vertex-Edge Graphs in Prekindergarten through Grade 12

Vertex-edge graphs comprise another discrete mathematics topic that is recommended in *Principles and Standards*. Vertex-edge graphs are mathematical models consisting of points (vertices) with curves or line segments (edges) connecting some of the points. Such diagrams can be used to solve problems related to paths, circuits, and networks. For example, vertex-edge graphs can be used to help optimize a telecommunications network, plan the most efficient circuit through cities visited by a salesperson, find an optimal path for plowing snow from city streets, or find the shortest route for collecting money from neighborhood ATM machines.

More abstractly, vertex-edge graphs may be useful when analyzing situations that involve relationships among a finite number of objects – the objects are represented by vertices and the relationship among the objects is shown by connecting some vertices with edges. The relationship may be very concrete, such as how cities are connected by airline routes in the salesperson example above, or they could be more abstract, like “conflict” or “prerequisite.” For example, you can use a vertex-edge graph to schedule committee meetings without conflicts (where two committees that conflict because of a shared member are linked by an edge), or to find the earliest completion time for a large construction project consisting of many tasks (where directed edges are used to link a task to its prerequisite tasks).

The formal study of vertex-edge graphs is called *graph theory*. The term *vertex-edge graph* is used to distinguish these diagrams from other types of graphs, like graphs of functions, or graphs used in data analysis such as bar graphs. Nevertheless, a commonly used term is simply *graphs*. We will use both terms, as appropriate.

Graph theory is part of the field of discrete mathematics, but it can also be thought of as part of geometry since graphs are geometric diagrams consisting of vertices and edges. Graphs share some characteristics with other geometrical objects in school mathematics, for example, both polyhedra and graphs have vertices and edges. But in contrast to most of school geometry, in which we are concerned with the size and shape of figures, size, shape, and position are not essential characteristics of vertex-edge graphs. When solving a graph problem, it doesn't really matter if the graph is large or small or if the edges are straight or curved. All that really matters are the numbers of vertices and edges and how the vertices are connected by the edges.

There are several fundamental graph theory topics that should be part of the school mathematics curriculum. The table below provides a brief summary of these.

Optimal Paths and Circuits

Graph Topic	*Basic Problem*	*Sample Application*
Euler paths	Find a route through a graph that uses each edge exactly once.	Determine snow plow routes.
Hamilton paths	Find a route through a graph that visits each vertex exactly once.	Rank tournament players.
Shortest paths	Find a shortest path from here to there.	Measure degree of influence among people in a group
Critical paths	Find a longest path/critical path.	Schedule large projects.
Traveling Salesman Problem (TSP)[1]	Find a circuit through a graph that visits all vertices, start = end, and has minimum total weight.	Determine the least expensive circuit through cities visited by a sales representative.

[1] Although sales representatives are nowadays as likely to be female as male, back when this problem was formulated, that was not the case, so the historic name for this problem is the Traveling Salesman Problem.

Optimal Spanning Networks

Graph Topic	*Basic Problem*	*Sample Application*
Minimum spanning trees	Find a network within a graph that joins all vertices, has no circuits, and has minimum total weight.	Create an optimal computer or road network.

Optimal Graph Coloring

Graph Topic	*Basic Problem*	*Sample Application*
Vertex coloring	Assign different colors to adjacent vertices, using the fewest number of colors.	Avoid conflicts – e.g., meeting schedules or chemical storage.

The analysis and representation of all these problems is very concrete at the early grades and becomes more formal and abstract as one moves upward through the grades. At all grade levels, students should:

- Use vertex-edge graphs to model and solve a variety of problems related to paths, circuits, networks, and relationships among a finite number of objects
- Understand and apply properties of graphs
- Devise, describe, and analyze algorithms to help solve problems related to graphs
- Use graphs to understand and solve optimization problems

Key themes at all grades are mathematical modeling, applications, optimization, and algorithmic problem solving. Mathematical modeling is a multi-step process of solving a real-world problem by using mathematics to represent the problem, finding a mathematical solution, translating that solution into the context of the original problem, and, finally, interpreting and judging the reasonableness of the result. Optimization problems are important throughout mathematics and in many applications. The goal is to find the best solution, for example, the shortest path, the most efficient strategy, the fewest conflicts, or the earliest completion time. Algorithmic problem solving is the process of devising, using, and analyzing algorithms (step-by-step procedures) for solving problems.

When teaching all these vertex-edge graph topics and themes, it is important not to get bogged down in the formality of definitions and algorithms. The visual nature of graphs should be used to make this material engaging, accessible, and fun. In fact, if vertex-edge graphs are presented in this way, many students who may have previously experienced difficulty or apathy in mathematics will find that the study of graphs is refreshing and interesting, they will experience success in learning this topic, and thereby gain confidence as they dig into other topics.

The following recommendations suggest how vertex-edge graphs can be developed throughout the grades.

Recommendations for Vertex-Edge Graphs in Grades PreK–12

In prekindergarten through grade 2 all students should:

- Build and explore vertex-edge graphs using concrete materials
- Explore simple properties of graphs, such as the numbers of vertices and edges, neighboring vertices and the degree of a vertex, and whole-number weights on edges
- Use graphs to solve problems related to paths, circuits, and networks in concrete settings
- Color simple pictures using the fewest number of colors
- Follow and create simple sets of directions related to building and using graphs
- Concretely explore the notion of the shortest path between two vertices

In grades 3–5 all students should:

- Draw vertex-edge graphs to represent concrete situations
- Investigate simple properties of graphs, like the degree of a vertex, weights on edges, and how to physically manipulate two graphs to determine if they are the "same"

- Use graphs to solve problems related to paths, circuits, and networks in concrete and abstract settings
- Color maps and color the vertices of a graph using the smallest number of colors, as an introduction to the general problem of avoiding conflicts
- Follow, devise, and describe step-by-step procedures related to working with graphs
- Analyze graph-related problems in terms of finding the “best” solution

In grades 6–8 all students should:

- Represent concrete and abstract situations using vertex-edge graphs, and represent vertex-edge graphs with adjacency matrices
- Describe and apply properties of graphs, such as degree of vertex, directed edges, edge weights, and whether two graphs are the “same” (isomorphic)
- Use graphs to solve problems related to paths, circuits, and networks in real-world and abstract settings, including explicit use of Euler paths, Hamilton paths, minimum spanning trees, and shortest paths
- Understand and apply vertex coloring to solve problems related to avoiding conflicts
- Use algorithmic thinking to solve problems related to vertex-edge graphs
- Use vertex-edge graphs to solve optimization problems

In grades 9–12 all students should:

- Study the following key topics related to vertex-edge graphs: Euler paths, Hamilton paths, the Traveling Salesman Problem (TSP), minimum spanning trees, critical paths, shortest paths, vertex coloring, and adjacency matrices
- Compare and contrast topics, in terms of algorithms, optimization, properties, and types of problems that can be solved
- Understand, analyze, and apply vertex-edge graphs to model and solve problems related to paths, circuits, networks, and relationships among a finite number of elements, in real-world and abstract settings
- Devise, analyze, and apply algorithms for solving vertex-edge graph problems
- Extend work with adjacency matrices for graphs, such as interpreting row sums and using the nth power of the adjacency matrix to count paths of length n in a graph

Overview of Iteration and Recursion in Prekindergarten through Grade 12

Iteration and recursion is the third main discrete mathematics topic recommended in *Principles and Standards*. Iteration and recursion are powerful tools for representing and analyzing regular patterns in sequential step-by-step change, like day-by-day change in chlorine concentration in a swimming pool, year-by-year growth of money in a savings account, or the total cost of postage as the number of ounces increases.

To iterate means to repeat, so iteration is the process of repeating the same procedure or computation over and over again, like adding 4 each time to generate the next term in the sequence 4, 8, 12, 16, … . Recursion is the method of describing a given step in a sequential process in terms of previous steps. You can often use a recursive formula to describe an iterative process. For example, you could describe the pattern in the sequence above with the recursive formula NOW = PREVIOUS + 4 or $s_n = s_{n-1} + 4$. (The cluster s_n of symbols, read "s sub n," provides a name for an arbitrary term, the n'th one, of sequence s. Thus s_4 would be the 4^{th} term, s_{10} would be the 10^{th} term, etc. The second equation above indicates that the n'th term of the sequence is 4 more than the previous, or n-1'st term of the sequence; this is the same as what is indicated by the first equation.)

Thus, iteration and recursion are like two sides of the same coin. You can think of recursion in terms of spiraling backward from current to previous steps, while iteration moves forward from the initial step. Both are powerful tools for analyzing regular patterns of sequential change. (Note that in computer science there are precise technical definitions for iteration and recursion, but we will use the terms here in the more informal sense just described.)

As with other topics, the work with iteration and recursion in the early grades is very concrete and exploratory. The representation and analysis become more abstract and formal as the grades progress. For example, in grades pre-K–2 students should explore sequential patterns using physical, auditory, or pictorial representations, like a pattern of hand claps that increases by two each time. In grades 3-5, students might describe a pattern of adding two each time as NEXT = NOW + 2. In middle school, students can begin using subscripts in a very basic way to describe patterns, such as describing the add-two pattern with the recursive formula $T_{n+1} = T_n + 2$. In high school, students can take a recursive view of functions, recognizing that, for example, NEXT = NOW + 2 could represent a linear function with slope 2.

In middle and high school, students should also compare and contrast recursive and explicit (or closed-form) formulas. For example, the sequence 5, 8, 11, 14, 17, … can be described by the recursive formula $s_n = s_{n-1} + 3$, with the initial term $s_0 = 5$, or by the explicit formula $s_n = 5 + 3n$, for $n \geq 0$. The recursive formula describes the step-by-step change, and gives a formula for the current term s_n in terms of the previous term s_{n-1}, while the explicit formula gives a formula for any term in the sequence s_n in terms of n. Some relative merits of these two representations are that the recursive formula more clearly shows the pattern of "add three each time," while the explicit formula is more efficient if you want to compute a term far along in the sequence, such as s_{50}.

The following recommendations suggest how iteration and recursion can be developed across the grades.

Recommendations for Iteration and Recursion in Grades Pre-K–12

In prekindergarten through grade 2, all students should:

- Describe, analyze, and create a variety of simple sequential patterns in diverse concrete settings
- Explore sequential patterns using physical, auditory, and pictorial representations
- Use sequential patterns and iterative procedures to model and solve simple concrete problems
- Explore simple iterative procedures in concrete settings using technology, such as LOGO-like environments and calculators

In grades 3–5, all students should:

- Describe, analyze, and create a variety of sequential patterns, including numeric and geometric patterns (such as repeating and growing patterns, tessellations, and fractal designs)
- Represent sequential patterns using informal notation and terminology for recursion, such as NOW, NEXT, and PREVIOUS
- Use sequential patterns, iterative procedures, and informal notation for recursion to model and solve problems, including in simple real-world contexts such as growth situations
- Describe and create simple iterative procedures using technology, such as LOGO-like environments, spreadsheets, and calculators.

In grades 6–8, all students should:

- Describe, analyze, and create simple additive and multiplicative sequential patterns (in which a constant is added or multiplied at each step), and more complicated patterns, such as Pascal's triangle (in which, except for the first two rows, each row of numbers is constructed from the previous row) and the Fibonacci sequence: 1, 1, 2, 3, 5, 8, … (in which the previous two terms are added to get the current term)
- Use iterative procedures to generate geometric patterns, including fractals like the Koch snowflake and Sierpinski's triangle
- Use informal notation such as NOW and NEXT, as needed, and subscript notation to represent sequential patterns
- Find and interpret explicit (closed-form) and recursive formulas for simple additive and multiplicative sequential patterns, and translate between these formulas
- Use iterative procedures and simple recursive formulas to model and solve problems, including in simple real-world settings
- Describe, create, and investigate iterative procedures using technology, such as LOGO-like environments, spreadsheets, calculators, and interactive geometry software

In grades 9-12, all students should:

- Describe, analyze, and create arithmetic and geometric sequences and series
- Create and analyze iterative geometric patterns, including fractals with an investigation of self-similarity and the areas and perimeters of successive stages
- Represent and analyze functions using iteration and recursion
- Use subscript and function notation to represent sequential patterns
- Investigate more complicated recursive formulas, such as simple non-linear formulas, formulas in which the added quantity is a function of n (like $S(n) = S(n-1) + (2n+1)$), and formulas of the form $A(n+1) = r\,A(n) + b$ (recognizing that when $r = 1$ the resulting sequence is arithmetic, and when $b = 0$ the resulting sequence is geometric)
- Use the method of finite differences to find explicit (closed-form) formulas for sequences that can be represented by polynomial functions

- Understand and carry out proofs using mathematical induction, recognizing a typical situation for induction proofs in which a recursive relationship is known and used to prove an explicit formula
- Use iteration and recursion to model and solve problems, including in a variety of real-world settings, and particularly as related to applied growth situations such as population growth and compound interest
- Describe, analyze, and create iterative procedures and recursive formulas using technology, such as computer software, graphing calculators, and programming languages

Conclusion

Three key topics of discrete mathematics are recommended in *Principles and Standards*: combinatorics (systematic listing and counting), vertex-edge graphs, and iteration and recursion. This article has presented an overview of these topics and specific recommendations for how they can be developed across the grades from pre-kindergarten through grade 12. The NCTM *Navigations* books on discrete mathematics, the introduction of which is directly adapted for this article, elaborate on each of these topics and provide many classroom activities.

Students need to understand and be able to apply discrete mathematics to be competitive as adults in our fast-changing, technology-rich, information-dense world. Discrete mathematics is increasingly "the math for our time" (Dossey, 1990). Discrete mathematics topics are engaging, contemporary, and useful. They should be part and parcel of today's school mathematics. As recommended in *Principles and Standards*, "discrete mathematics should be an integral part of the school mathematics curriculum" (p. 31). By meeting this recommendation we can bring the power of discrete mathematics to all students in all grades.

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Eric has written textbooks and articles, directed several NSF teacher enhancement projects and an NCTM online materials development project, and worked extensively on state department of education projects in his home state of Iowa.

References

Consortium for Mathematics and Its Applications (COMAP). *Mathematics: Modeling Our World.* New York, NY: W. H. Freeman, 1998-2000.

_____________. *For All Practical Purposes*, 6th ed. New York, NY: W.H. Freeman and Company, 2003.

Crisler, Nancy and Gary Froelich. *Discrete Mathematics Through Applications, 3rd ed.,* New York: W.H. Freeman and Company, 2005.

DeBellis, V.A. & Rosenstein, J.G., Making Math Engaging: Discrete Mathematics for K-8 Teachers. Greenville, NC: Javelando Publications, 2008.

Dossey, John A. "Discrete Mathematics: The Math For Our Time." In *Discrete Mathematics Across the Curriculum K-12*, 1990 Yearbook of the National Council of Teachers of Mathematics, edited by Margaret J. Kenney and Christian R. Hirsch. Reston, Va.: NCTM, 1990.

Hirsch, Christian R., James T. Fey, Eric W. Hart, Harold L. Schoen, and Ann E. Watkins. *Core-Plus Mathematics.* 2nd. ed., Courses 1 – 4. New York: Glencoe, 2008–09.

Mathematical Association of America (MAA). *Report of the Committee on Discrete Mathematics in the First Two Years*. Washington, D. C.: MAA, 1986.

National Council of Teachers of Mathematics (NCTM). *Discrete Mathematics and the Secondary Mathematics Curriculum*. Reston, Va.: NCTM, 1990.

___________. *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

___________. *Principles and Standards for School Mathematics*. Reston, Va.: NCTM, 2000.

Prensky, Marc (2001). Digital natives, digital immigrants. *On the Horizon*. NCB University Press Vol. 9 No. 5 (October 2001).

Rosenstein, Joseph G., Deborah S. Franzblau, and Fred S. Roberts (Eds.). *DIMACS Series in Discrete Mathematics and Theoretical Computer Science: Vol. 36. Discrete Mathematics in the Schools.* Providence, RI: American Mathematical Society & National Council of Teachers of Mathematics, 1997.

Tannenbaum, Peter. *Excursions in Modern Mathematics,* 6th ed. Upper Saddle River, NJ: Pearson Prentice Hall, 2007.