GENERALIZATION OF SOME OPTIMIZATION PROBLEMS.

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Abstract. In this paper I solve several optimization problems in the calculus books by showing that the maximal area of a rectangle with vertices on the side of any triangle always equals one half of the area of the triangle. I also show that the ratio between the largest area of a rectangle with two vertices on a horizontal line and two vertices on a parabola, and the area between the parabola and the horizontal line is always constant and equals $\frac{1}{\sqrt{3}}$, and that the rectangle with the maximum area is always obtained by cutting the parabola at a constant height which is $\frac{2}{3}$ of the height of the parabola. I show similar results for cylinder inside a cone and cylinder inside a paraboloid.

Calculus students are familiar with problems of the following type: find the largest area of a rectangle inscribed in a triangle (i.e. all vertices are on the sides). For examples of such problems see [1] and [2] or just about any Calculus textbook. Usually the triangles are right triangle, or they have a vertex at the origin, e.t.c. What we propose here is to solve all these problems at once by using elementary mathematics accessible to any calculus student and look into some other similar problems. First, we make a simple observation that for any triangle, we can inscribe at least one rectangle with two vertices one side and the other two vertices on the remaining sides. The side with two vertices is the largest one on the triangle to make sure that such a rectangle exists. Second, any triangle can be regarded via translations and rotations as a triangle with one vertex at the origin, one on the x-axis and the third one in the first quadrant. The following surprising result is obtained.

Result.

Let ABC be a triangle with $A = (0, 0)$, $B = (b, 0)$ and $C = (c, h)$. Assume that $b \geq c$. The largest area of a rectangle that can be inscribed in the triangle ABC, with one side of the rectangle on the $x-axis$, equals one half of the area of the triangle.

Simple calculations show that any horizontal line $y=t$ intersects the lines AC and CB at the points $(\frac{ct}{h},t)$ and $(b+\frac{t(c-b)}{h},t)$ respectively. Therefore our rectangle has length

$$l=b+\frac{t(c-b)}{h}-\frac{ct}{h}=b-\frac{tb}{h} \text{ and width } w=t.$$

Hence the area of the rectangle as a function of the parameter t is given by

$$A(t)=(b-\frac{tb}{h})t,$$

which is a quadratic function whose maximum is attained at the vertex $(\frac{h}{2},\frac{bh}{4})$. This finishes the argument, since the area of the triangle is $\frac{bh}{2}$.

Since we love to find patterns, it is natural to think that a similar pattern holds for similar optimization problems and, as we will see, it does. For any region under a parabola (i.e. the region formed by a parabola and a line perpendicular to the line of symmetry) the rectangle with the largest area that can be inscribed in the region under the parabola, depends only on the height of the region. A helpful observation is that any parabola region can be regarded as a open down parabola with the x-intercepts at the origin and at the point $(a,0)$ and with vertex of coordinates $(\frac{a}{2},-\frac{ma^2}{4})$.

Another surprising result

For any region under a parabola, the rectangle with the largest area that can be inscribed in the region is obtained by drawing the horizontal line $y=-\frac{ma^2}{6}$. The height of the cut is always two thirds of the height of the parabola and, the ratio between the area of the rectangle and the area of the parabola region is constant $\frac{1}{\sqrt{3}}$

Let $y=mx(x-a)$ be our quadratic function where m is a negative real number. Then the area of the parabola region is $\int_0^a mx(x-a)dx=-\frac{ma^3}{6}$. For $c\in(0,-\frac{ma^2}{4})$, the equation $mx(x-a)=c$ has two positive real roots given by

$$x_{1,2}=\frac{a}{2}\pm\sqrt{\frac{c}{m}+\frac{a^2}{4}}.$$

Therefore, the width of the rectangle obtained is $w=|x_2-x_1|=2\sqrt{\frac{c}{m}+\frac{a^2}{4}}$ while its height (length) is $l=c$. Hence, the function we want to maximize is given by $A(c)=2c\sqrt{\frac{c}{m}+\frac{a^2}{4}}$. A simple calculus exercise shows that this function has a maximum at $c=-\frac{ma^2}{6}$ and that the maximum value of A equals $-\frac{ma^3}{6\sqrt{3}}$, which finishes the proof of the result.

Remark.

It is worth while to notice in here that for $c=-\frac{ma^2}{6}$ it follows that

$$x_{1,2}=\frac{a}{2}\pm\sqrt{\frac{c}{m}+\frac{a^2}{4}}=\frac{a}{2}\pm\frac{a}{2\sqrt{3}},$$

which is independent on m,i.e. for any parabola with $x-intercepts$ at 0 and a we get the largest rectangle simply by constructing two vertical lines at

$x_{1,2}=\frac{a}{2}\pm\frac{a}{2\sqrt{3}}$.

Looking at these two results, a natural question is if similar results will hold for different problems. Surprisingly, we got similar results for some optimization problems involving volumes.

Even more pleasant surprises!

1. For any right circular cone, the cylinder of largest volume that can be inscribed in the cone is always obtained by sectioning the cone at a height of $\frac{2}{3}$ of the height of the cone and the ratio between the two volumes is always constant $\frac{4}{9}$.

2. Let $y=m(x^2-a^2)$ be a parabola with $x-intercepts$ at $x=\pm a$, and height $-ma^2$. Rotating this parabola over the $y-axis$ we obtain a paraboloid. Then, the cylinder with the largest volume that can be inscribed in this paraboloid is obtained by sectioning the paraboloid at a height of $-\frac{ma^2}{2}$ which represent a ratio of $\frac{1}{2}$ from the height of the paraboloid. Moreover, the ratio between the volume of the largest

cylinder and the paraboloid is always constant $\frac{1}{2}$.

Using the method of washers or the method of cylindrical shells one computes the volume of the paraboloid as $V_p = -\frac{m\pi a^4}{2}$. Using a similar method with the one in the first result one computes the volume of the largest cylinder inside this paraboloid and obtains $V_c = -\frac{m\pi a^4}{4}$ and the proof is complete.

As a final remark, it seems that there are interesting patterns in at least some of the optimization problems in the calculus books that a student will be able to explore. I hope that these results will rise the curiosity of the calculus students to look for generalizations of the problems they are so often asked to solve.

References

1. Edwards & Penney, *Single Variable Calculus.* Prentice Hall **sixth edition** (2002), 227–250.
2. Varberg, Purcell & Rigdon, *Calculus.* Prentice Hall **ninth edition** (2006), 174-178.

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