Synthetic Division Revisited

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Abstract. Polynomial division for polynomials with rational coefficients (uaually integers) is a subject taught in most secondary school and college algebra level curricula. A survey of five popular textbooks used to teach this subject shows that these texts contain the same algorithm for this operation: an algorithm that mimics the long division algorithm taught in elementary school. In addition, each mentions that under a special set of circumstances (divisor is first order) that a more efficient algorithm may be used called synthetic division; none of these texts even hint that the synthetic division algorithm need not be limited by these constraints.

Following the work of Lanczos (1956) we show how the typical synthetic division algorithm can be used to perform almost all of the polynomial division problems encountered in secondary schools.

INTRODUCTION

Currently in the United States secondary school algebra and college algebra are dominated by expressions that are polynomials in one variable, typically with rational coefficients. Treating polynomials in one variable as algebraic objects is one of the more abstract parts of secondary school algebra. Students learn to add, subtract, and multiply these polynomials as part of their early experiences in a typical algebra course. In these cases the sum, difference, and product of two polynomials is again a polynomial.

Of course, the quotient of one polynomial by another isn't always a polynomial. In general quotients of polynomials with rational coefficients are rational functions (with rational coefficients). In this paper we consider five well-known texts used as bases for secondary school and college algebra instruction to see how they treat polynomial division. We find that each present two algorithms for computing polynomial divisions (long division and synthetic division), but that, in each case, the texts limit the synthetic division algorithm unnecessarily.

Finally, we provide a more complete picture of the synthetic division algorithm and provide examples that illustrate how to use the more complete algorithm. This work can be found in *Applied Analysis* by Cornelius Lanczos, Dover Press, 1956.

CURRENT CURRICULUM

Looking at current texts (*Algebra 2*, *Algebra & Trigonometry*, *College Algebra*, *College Algebra & Trigonometry*, *Intermediate Algebra*) we see that all teach polynomial division analogously to the way that integer division is taught to elementary school students. We begin by looking at the highest term of the divisor and finding the monomial which when multiplied by this term gives the highest power term in the dividend. This monomial is then multiplied by the divisor and the result is subtracted from the divisor. The result of this subtraction then becomes the “new” divisor (of smaller degree because the previous step was designed to remove the highest degree term) and the process is repeated until the degree of the “new” dividend is smaller than that of the divisor and what is left is the remainder.

Of course these texts provide examples for students to imitate and (like in elementary school), if students keep terms lined up correctly, most students find that they can master this algorithm. This author has observed numerous secondary and college level mathematics teachers demonstrating this technique to their students and in each case the students are asked to recall how they learned to divide whole numbers as a way of helping them learn this method.

In *Algebra 2* the reader is then introduced to synthetic division (sometimes known as Ruffini's Rule):

> "A simpler process called **synthetic division** has been devised to divide a polynomial by a binomial."

Of course, a systematic transfer of the pieces of the division algorithm to the tableau of synthetic division is presented. In some cases (*College Algebra*) the calculations for the two algorithms are presented side-by-side so that students can see the connection between the two algorithms. We can see that synthetic division is simply a computational algorithm when we read that students are reminded that (*Intermediate Algebra*):

> "When using synthetic division, if there are missing powers of the variable, insert 0s as coefficients."

In fact, in Sullivan & Sullivan (*Algebra & Trigonometry*) the reader is told that synthetic division makes the task [of polynomial division] simpler and that the process arises from:

> "rewriting the long division [algorithm] in a more compact form, using simpler notation."

From each of these sources we see that synthetic division is (1) a computational algorithm and (2) that it applies to binomial divisors of the form $ax+b$ which for the purpose of the given algorithm must be rewritten as $a(x-r)$ and the division by the constant a is done before the synthetic division algorithm is executed.

SHORT TERM LOGIC

The need for a more compact and straightforward computational tool for computing the quotient of two polynomials is clearly desirable if you have ever had to use the traditional division algorithm to perform this task. However, limiting the algorithm to divisors that are binomials isn't necessary.

One reason that synthetic division stops with binomial divisors and in fact divisors of the form $x-r$ is the need for this type of division when using the Remainder Theorem to factor and evaluate polynomials. In these schemes we use synthetic division to rewrite a polynomial, $P(x)$, as

$$P(x) = Q(x)(x - r) + R$$

where $Q(x)$ is the quotient of $P(x)$ when divided by $x - r$ and R is the remainder (a number in this case). It is easy to see that this allows us to evaluate $P(r) = R$. So the remainder in the synthetic division process is the value of the polynomial when $x = r$.

Of course this is helpful when one tries to locate the zeroes of a polynomial function—giving a simple algorithm to "search" for zeroes and for finding points that are on the graph of the function $y = P(x)$. But why stop there? What if we wanted to compute $\frac{x^6 - 1}{x^2 - 1}$? Should we have to factor the divisor and perform synthetic division twice? Below we display an "algorithm" that would allow us to easily compute the answer $\frac{x^6 - 1}{x^2 - 1} = x^4 + x^2 + 1$.

A More Complete Algorithm

On page 13, Lanczos notes that in certain calculations "a great deal of time is lost by noting down partial results which could have been avoided by a more concise arrangement of the calculations." He then describes one particular type of arrangement called a *movable strip*. In this arrangement one set of numbers is written in a column down a vertical strip of paper and the numbers in this column operate on numbers in another column called the *fixed strip*. The operation is to multiply two numbers facing each other as follows: the lowest element of the movable strip is multiplied by the corresponding element in the fixed strip and all other elements of the movable strip are multiplied by corresponding elements of the nascent strip. Finally, we add these partial products and write the result in what is called the *nascent strip.*

Once a calculation is completed the result is written in the nascent strip and the movable strip moves down one position and the calculations are repeated. The operation ends when the bottom of the movable strip reaches the bottom of the fixed strip.

To illustrate this methods applicability to polynomial division let's consider the division problem $\dfrac{2x^2-3x+5}{x+2}$. Synthetic division can then be recast in terms of movable, fixed, and nascent strips as follows. We write the coefficients of the dividend in a vertical column starting with the highest power term coefficient (fixed strip), write the coefficients of the divisor in a column starting with the highest power term coefficient (which must be normalized to 1) and moving upward, reversing the sign of each coefficient after the first (movable strip).

Movable Strip	*Nascent Strip*	*Fixed Strip*
-2		
1		
↓		2
		-3
		5

Moving the strip down one the first entry in the nascent strip is: (1)(2) = 2.

Movable Strip	*Nascent Strip*	*Fixed Strip*
-2		
1	2	2
↓		-3
		5

Moving the strip down once again we get the second entry in the nascent strip: (1)(-3) + (-2)(2) = -7

Movable Strip	*Nascent Strip*	*Fixed Strip*
-2	2	2
1	-7	-3
↓		5

The final moving of the movable strip down computes the last entry of the nascent strip: (1)(5) + (-2)(-7) = 19, as shown below:

Movable Strip	*Nascent Strip*	*Fixed Strip*
	2	2
-2	-7	-3
1	19	5

For those familiar with synthetic division we see that the coefficients of the quotient are the first two entries computed in the nascent strip and that the remainder is the last entry computed. Hence we can read off the calculation as

$$\frac{2x^2 - 3x + 5}{x + 2} = 2x - 7 + \frac{19}{x + 2}.$$

For a small bit of work (in this case having to add the two partial products) we get the same result as the traditional "synthetic division" algorithm. Moreover, if students are taught by first making movable strips on small pieces of paper and aligning the terms that comprise the partial products the instruction could appeal to more visually stimulated learners.

Most importantly, this technique works regardless of the degree of the divisor! We only have to make sure that the lead coefficient of the divisor is 1. As a second example let's consider the problem of computing the division:

$$\frac{x^4 - 5x^3 + 2x - 2}{x^2 + 3}.$$

We begin by writing the appropriate strips for doing the computation:

Movable Strip	*Nascent Strip*	*Fixed Strip*
-3		
0		
1		
↓		1
		-5
		0
		2
		-2

In this case we simply show the results of the movement of the strip to the bottom of the fixed column.

Movable Strip	*Nascent Strip*	*Fixed Strip*
	1	1
	-5	-5
-3	-3	0
0	17	2
1	7	-2

We may now read the calculation of the division as:

$$\frac{x^4 - 5x^3 + 2x - 2}{x^2 + 3} = x^2 - 5x - 3 + \frac{17x + 7}{x^2 + 3}.$$

Lanczos notes that the process can be made more “friendly” for polynomial division by writing the names of the strips and adding a column for the remainder as follows:

Divisor	*Quotient*	*Dividend*	*Remainder*
	1	1	
	-5	-5	
-3	-3	0	
0		2	17
1		-2	7

Here we must understand that the remainder will begin when we have reached a position in the dividend where the term indicated is one degree less than the degree of the divisor.

As a last example let's try to compute the quotient: $\dfrac{3x^7+9x^4+x^3}{x^5-5x+4}$.

We begin with the table:

Divisor	*Quotient*	*Dividend*	*Remainder*
-4			
5			
0			
0			
0			
1			
↓		3	
		0	
		0	
		9	
		1	
		0	
		0	
		0	

Computing using the movable strip gives us:

Divisor	*Quotient*	*Dividend*	*Remainder*
	3	3	
	0	0	
-4	0	0	
5		9	9
0		1	16
0		0	-12
0		0	0
1		0	45

Which you can readily check gives us:

$$\frac{3x^7 + 9x^4 + x^3}{x^5 - 5x + 4} = 3x^2 + \frac{9x^4 + 16x^3 - 12x^2 + 45}{x^5 - 5x + 4}.$$

As this method is iterative, it is suitable for implementation a hand-held graphing calculator like the TI-83/4 family of calculators. Below is an algorithm for implementing this method on these calculators:

PROGRAM:POLYDIV

: 0 → D : 0 → E (*Initialize dimensions of dividend (D) and divisor (E)*)

: ClrHome

: Lbl 1

: Disp “INPUT DEGREE”

: Disp “OF DIVIDEND”

: Prompt D (*Get degree of dividend*)

: D + 1 → dim(ÙDVDND)

: Disp “INPUT COEFFS”

: Disp “OF DIVIDEND”

: For(I, 1, D+1) (*Get dividend coefficients from highest power to lowest)*

: Input X

: X → ÙDVDND(I)

: End

: Disp ÙDVDND → Frac (*Echo dividend to check for correct input*)

: 0 → A

: Disp “CHANGE(1=YES)

: Input A

: If A = 1

: Goto 1

: Lbl 2

: Disp “INPUT DEGREE” (*Get degree of divisor*)

: Disp “OF DIVISOR”

: Prompt E

: E + 1 → dim(ÙDVSR)

: Disp “INPUT COEFFS”

: Disp “OF DIVISOR”

: For(J, 1, E+1) (*Get divisor coefficients from highest power to lowest*)

: Input X

: X → ÙDVSR(J)

: End

: Disp ÙDVSR → Frac (*Echo divisor to check for correct input*)

: 0 → A

: Disp “CHANGE(1=YES)

: Input A

: If A = 1

: Goto 2

: D+1-E → dim(ÙQTNT) (*Compute size of quotient whose degree is D—E*)

: E → dim(ÙRMDR)

: ÙDVDND/ÙDVSR(1) → ÙDVDND (*Divide dividend by divisor’s lead coefficient*)

: ÙDVSR/ÙDVSR(1) → ÙDVSR (*Divide divisor by divisor’s lead coefficient*)

: -ÙDVSR → ÙDVSR (*For algorithm reverse sign of divisor’s coefficients*)

: 1 → ÙDVSR(1) (*Make lead coefficient of divisor 1*)

: For(K, 1, D+1) (*Begin moving movable strip through D+1 terms of dividend)*

: min(K, E+1) → M (*Only do calculations for overlap of strips*)

: 0 → S

: For(L, 1, M)

: S + ÙDVDND(K+1-L)* ÙDVSR(L) → S (*Compute partial sums*)

: End

: S → ÙDVDND(K) (*Replace dividend entry with nascent strip calculation*)

: End

: ClrHome

: For(I, 1, D+1-E)

: ÙDVDND(I) → ÙQTNT(I) (*Break nascent strip into Quotient and Remainder*)

: End

: For(I, 1, E)

: ÙDVDND(D+1-E+I) → ÙRMDR(I)

: End

: Disp "QUOTIENT"

: Disp ÙQTNT → Frac (*Displace coefficients of Quotient*)

: Disp "REMAINDER"

: Disp ÙRMDR → Frac (*Displace coefficients of Remainder*)

: Stop

This program may be obtained by emailing atalmadg@uno.edu

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