Trisection using Bisection

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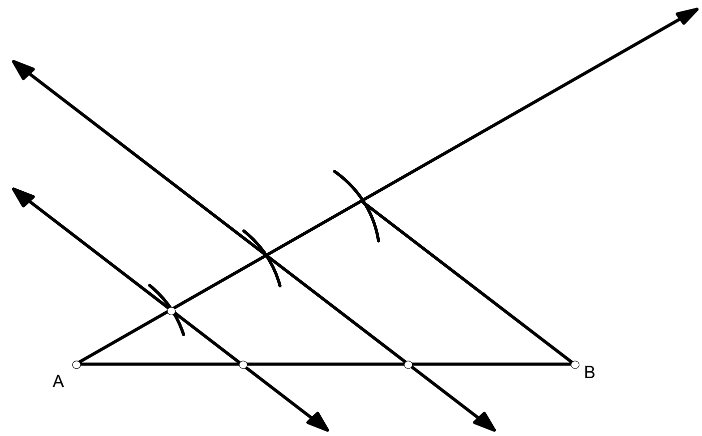
Abstract.
It is possible to trisect a line by using only bisections. In this article we describe the algorithm and offer two different proofs of its validity. We also generalize the process to partitioning a segment into n equal parts.

Both authors have had the pleasure of teaching the traditional junior/senior level College Geometry course. The first part of the first author's semester discussed non-Euclidean geometries and working with axiomatic systems. The second part of the course discussed Euclidean geometry and introduced the Geometer's Sketchpad. This is where our story begins.

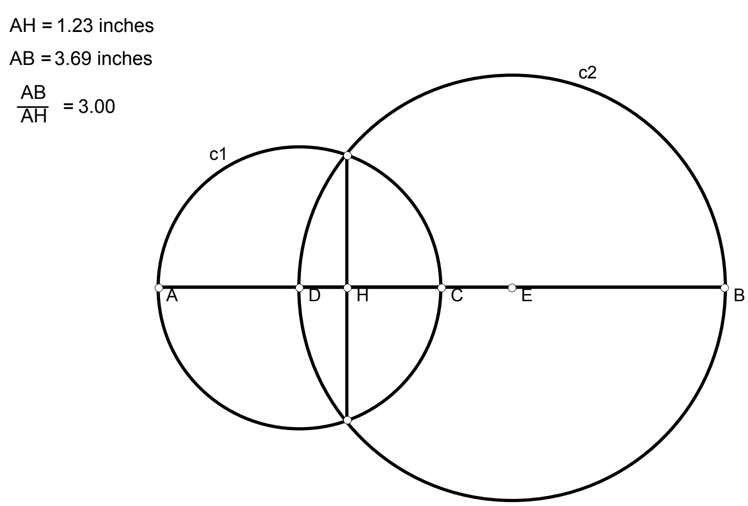
Most of the students in the class were future high school teachers and Geometer's Sketchpad is often used in high school with a discovery approach to learning geometry. We followed this discovery approach in this portion of the course so that the students would better understand their future students' perspective.

After the student had worked with Sketchpad for about a week, we assigned the following problem: Given a line segment, trisect this segment. We fully expected most of the students to use Euclid's construction published in most high school textbooks. That is, construct a ray emanating from one of the endpoints. Along this ray construct three equally spaced points. From the last point constructed on the ray, construct a line segment *l* to the other endpoint of the original line segment. Now construct lines parallel to *l* through the other two points constructed along the ray. This divides the original line segment into three equal pieces.

Euclid's method for trisecting a line.



While walking around the room I noticed two groups had the following figure.



Thinking that this was a special case, I reached down and clicked on one of the endpoints to change the length of a line segment. Much to my chagrin, the ratio on the screen did not change. According to the Sketchpad, they had indeed accomplished the task set forth.

Here is their construction:

1) Construct a midpoint **C** of the line segment $\overline{AB}$.

2) Construct a midpoint **D** of the line segment $\overline{AC}$.

3) Construct a midpoint **E** of the line segment $\overline{DB}$.

4) Construct a circle **c1** centered at **D** and radius $\overline{\mathbf{DA}}$.

5) Construct a circle **c2** centered at **E** and radius $\overline{\mathbf{EB}}$.

6) Construct a line segment from the two points where the circles intersect.

7) The intersection of this line segment with the original line segment forms a point exactly one-third the length of the line.

I was interested in the thought processes that led them to this construction. Was there a theorem they were thinking of? Perhaps they had seen another construction that made them think of this. Instead, they informed me that since Geometer's Sketchpad had a button for the midpoint, they thought that they needed to use that. Chalk another one up for serendipity.

After verifying for myself that the construction is valid, I asked my students if they could prove it themselves. The question became a homework assignment, but turned out to be difficult for the class. It came as little consolation that the proof could be obtained using only college algebra skills.

Algebraic Proof of Construction: Without loss of generality assume that point **A** has coordinates (0,0) and point **B** has coordinates (0,**b**). Then the equations of the circles are:

$$c1 \Rightarrow \left(x - \frac{b}{4}\right)^2 + y^2 = \left(\frac{b}{4}\right)^2$$

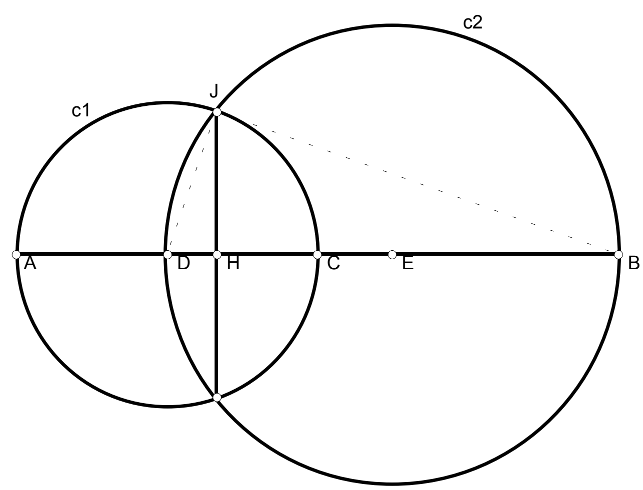
$$c2 \Rightarrow \left(x - \frac{5b}{8}\right)^2 + y^2 = \left(\frac{3b}{8}\right)^2$$

Solving these two equations simultaneously yields $\mathbf{x} = \frac{\mathbf{b}}{3}$. □

It should be noted that this was an excellent opportunity to point out to the class both the importance of having sound algebraic skills and the connection between algebra and geometry.

As this story was related to the second author, the question changed slightly. Could this result be generalized to *n*-secting a segment or does it depend on the fact that trisecting is the goal?

The first step to a generalization is to write the proof geometrically.



Geometric Proof of Construction: Use the notation in the diagram above and additionally label one of the points of intersection of the two circles **J**. Then triangles **DJB** and **DHJ** are similar right triangles. Thus, **DH**/**DA** = **DH**/**DJ** = **DJ**/**DB** = **DA**/**DB** = 1/3. Notice also that **DA**/**AB** = 1/4, by construction of **D**. If **DH** is taken as the unit length, then **DA** = 3, **AB** = 12 and **AH** = 4 whereby **AH**/**AB** = 1/3 as desired. ☐

The apparent coincidence between the values of **AH**/**AB** and **DA**/**DB** actually holds true in general because

$$
\begin{aligned}
\mathbf{AH}/\mathbf{AB} &= (\mathbf{DA}+\mathbf{DH})/\mathbf{AB} \\
&= \mathbf{DA}/\mathbf{AB} + \mathbf{DH}/\mathbf{AB} \\
&= \mathbf{DA}/\mathbf{AB} + (\mathbf{DH}/\mathbf{DA})*(\mathbf{DA}/\mathbf{AB}) \\
&= \mathbf{DA}/\mathbf{AB} + (\mathbf{DA}/\mathbf{DB})*(\mathbf{DA}/\mathbf{AB}) \\
&= (\mathbf{DA}/\mathbf{AB})*(1+\mathbf{DA}/\mathbf{DB})
\end{aligned}
$$

= (**DA**/**AB**)*(**DB**+**DA**)/**DB**
= (**DA**/**AB**)*(**AB**/**DB**)
= **DA**/**DB**.

Generalization: To *n*-sect a segment, note first that the case for $n = 2^m$ is simply repeated bisection.

In other cases, we start by choosing *m* such that $2^{m-1} < n < 2^m$ and use repeated bisection to find a segment $1/2^m$ of the original one. This corresponds to **AD** in the diagram, so label that point **D**. Copy that segment **AD** *n* times starting at **D** and in the direction of **B**. Label the end of these copies **K**, such that **D** is between **A** and **K** and **AD/DK** = 1/n. Note that in the case of trisection, **K** and **B** coincided, but in general this won't happen. Bisect **DK** giving **E**. We now have the centers of the two circles (**D** and **E**) and the construction proceeds as above to obtain **H**. The proof that **DH** is 1/*n* of **DA** is also as above.

At this point in the original proof, we moved from **DH** being 1/*n* of **DA** to **AH** being 1/*n* of **AB.** However, this will not be the case in general. To finish the general case, we need to copy segment **DH** 2^m-*n* times in the direction of **B**. This gives a new point **L**, such that **H** is between **D** and **L** and such that **DH**/**DL** = $1/(2^m-n)$. Then, using **DA** as the unit (so that **DH** = 1/*n* and **AB** = 2^m), we have **AL** = **AD** + **DL** = 1 + (2^m-n)**DH** = $1 + (2^m-n)/n = 2^m/n$. Thus, **AL**/**AB** = $(2^m/n)/(2^m) = 1/n$ as desired.

Patrick Mitchell received his Ph. D. from Kansas State University and has taught full-time for the last six years in Louisiana, Nebraska and Texas. Currently he is on the faculty at Midwestern State University.

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