What Is Problem-solving Ability?

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Abstract

This study addresses the question: "what is problem-solving ability?" and an attempt is made to answer the question from three points of view: *Principles and Standards for School Mathematics* (*Principles and Standards*, National Council of Teachers of Mathematics, 2000), a sample of mathematics educators and a sample of mathematicians. The *Principles and Standards* defines problem solving as "engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p. 52) and mathematics educators agree. Mathematicians are not concerned that the solution method is not known in advance, only that the solution method is not given in advance. The suggestion is given that if the solving of routine problems is overlooked, an important part of mathematics education is being missed.

Introduction

Who would disagree that it is important that mathematics students have problem-solving abilities? In fact, some NCTM-oriented curricula aim to produce problem-solving abilities in their students and claim to be more successful at that task than the traditional curricula (Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000; Schoen & Ziebarth, 1998). A body of research (Latterell, 2000) that has compared the problem-solving abilities of NCTM-orientated curricula students to traditional curricula students showed no statistically significant differences as measured by paper-and-pencil standardized tests. However, difficulty arose in this research over the question: Just what does "problem-solving ability" mean? The author also conducted a research study with these same two groups of students using an open-ended problem-solving test (videotaped, problems of a nonroutine nature). This time the NCTM-oriented curricula students significantly out-performed traditional students. Did the tests measure two different things? Is there more than one definition of problem solving?

Kantowski stated that "a problem is a situation for which the individual who confronts it has no algorithm that will guarantee a solution. That person's relevant knowledge must be put together in a new way to solve the problem" (1980, p. 195). Polya defined problem solving as finding "a way where no way is known, off-hand… out of a difficulty…around an obstacle" (1949/1980, p. 1). Polya stated that to know mathematics is to solve problems.

The difference between nonroutine and routine problems seems to be a key element in how problem solving is currently being viewed among mathematics educators. Carpenter (1988) emphasized that learning "a collection of problem-solving procedures" (p. 188) is not problem solving. Lester (1985) stated it this way:

> The primary purpose of mathematical problem-solving instruction is not to equip students with a collection of skills and processes, but rather to enable them to think for themselves. The value of skills and process instruction should be judged by the extent to which the skills and processes actually enhance flexible, independent thinking. (p. 66)

Dowshen (1980) conducted a critical analysis of the research on problem solving in secondary school mathematics between the years of 1925-1975. Out of twelve conclusions, one stated the following.

> Characteristics of an effective problem solver can be identified. An effective problem solver: tends to use a wide range of heuristic strategies; seems to follow some plan of attack when solving a problem and exhibits trial-and-error ability; has good arithmetic skills; has confidence in own mathematics ability; tends to check answers for reasonableness and is able to estimate an answer; and usually obtains an understanding of a problem before trying to solve it.

Research has also been conducted regarding what constitutes the process of problem solving. Polya (1945/1973) posited four problem-solving steps in *How to Solve It*: understanding the problem, devising a plan, carrying out the plan and looking back.

As obvious as this may seem, we should not take for granted that mathematics educators' views of problem solving are universally accepted. The question is posed "what is problem-solving ability?" and an attempt is made to answer the question from three different perspectives: 1) that given in *Principles and Standards for School Mathematics* (*Principles and Standards*, National Council of Teachers of Mathematics, 2000), 2) the views of a sample of mathematics educators, and 3) the views of a sample of mathematicians.

Principles and Standards is having considerable effect on secondary mathematics teachers. The President of NCTM describes *Principles and Standards* as "a guide to what mathematics students should learn; what teaching practices, approaches, and tools show promise; and the role that assessment plays in judging students' performance and the effectiveness of mathematics programs" (Stiff, 2001). *Principles and Standards* is intended to be "a resource and guide for all who make decisions that affect the mathematics education of students in prekindergarten through grade 12" (NCTM, 2000, p. ix). It is made up of the principles (six perspectives) and the standards (five content goals and five process goals).

The term "mathematics educator" is used in this study to mean someone holding a graduate degree in mathematics education (or a closely related field) and professionally active in some aspect of mathematics education. Mathematics educators sampled included primary authors of National Science Foundation (NSF)-funded curricula. In the 1990s, NSF sponsored the creation of 13 mathematics curricula programs for kindergarten through grade 12 for the purpose of having available curricula aligned with NCTM-standards (at that time, it was the 1989 *Curriculum and Evaluation Standards for School Mathematics*). Examples of curricula include Everyday Mathematics, TERC's Investigations in Number, Data, and Space, Connected Mathematics, Mathematics in Context, and Core-Plus. Such curricula are appearing with increasing frequency in secondary mathematics programs. Mathematics educators are in a position to influence the mathematical education of secondary students.

The sample of mathematicians were instructors of undergraduate mathematics courses. Mathematics education continues into the college years and a "seamless" transfer from secondary mathematics to undergraduate mathematics is a high priority. Mathematicians are in a position to influence the mathematical education of undergraduate students.

The answers to the question, "what is problem-solving ability?" has many implications. For example, the answer is clearly important when testing problem-solving ability, whether this is in standardized tests at the secondary level or in the process of placing incoming students into college mathematics courses. The answer is imperative when interpreting research comparing NCTM-oriented curricula to traditional curricula. If we are not clear on our definition of problem-solving ability, it is difficult to value or even interpret results that suggest a particular curricula produces problem-solving ability. The next section describes the method used to attempt to answer the question, "what is problem-solving ability?"

Method

The *Principles and Standards* (2000) was examined for a definition of problem solving, how problem solving was approached in the various grade levels, and sources of possible internal inconsistencies.

A list of mathematics educators was complied using the criteria of significant publications in the area of problem solving. To determine significantly published, an ERIC search was conducted using keywords of "mathematics" and "problem solving" and those publishing ten or more journal articles on problem solving were added to the list. No claim is made that this process was exhaustive or that the "best" list was formed. Certainly, the list did contain a subset of those mathematics educators with considerable expertise in problem solving. The list consisted of 13 mathematics educators who were each contacted and asked to participate in a study of problem solving. The eight who agreed to participate included authors of NSFfunded curricula. All those contacted were employed as professors of mathematics education and worked in positions that included research and teaching responsibilities.

A list of mathematicians was also compiled. The criteria for this list were more difficult to construct. The mathematics educators chosen had direct influence on the secondary mathematics curricula. For an equivalent level of influence, mathematics professors would have to teach the undergraduate mathematics courses as well as influence the curricula of the undergraduate program. Research by mathematicians includes various areas of pure and applied mathematics that are not related to mathematics education. Thus publication records were not as beneficial. Randomly selecting a professor of mathematics from a college or university would not necessarily result in someone who had the required influence on undergraduate mathematics courses.

It was decided to contact the participants in a professional development program for Ph.D.s in mathematics who desired to improve the teaching and learning of undergraduate mathematics. The program chosen was competitive (applications were required) and the participants attended workshops on alternatives to the traditional lecture approach to teaching mathematics. Topics such as reform calculus, active learning, group projects, incorporation of technology, writing across the curriculum, involving undergraduates in research, and research in education itself were covered. All participants in this program were contacted at the end of the official program and asked to participate in a study on problem solving. Of the 71 contacted, seven agreed to do so.

The participants (both the mathematics educators and the mathematicians) were sent a survey to fill out. The survey contained numerous questions of a Likert-scale variety as well as space for open-ended responses. The scale questions concerned issues regarding the measurement of problem solving, how various problem-solving items align with *Principles and Standards*, what it means to be nonroutine, problem-solving strategies and measurement of problem-solving concepts. Most important to this study was the question directly asking participants to define problem solving. The participants were asked to complete the survey and return it.

<u>Results</u>

Principles and Standards

Problem solving is one of ten standards occurring across all grade levels in the *Principles and Standards*. The authors define problem solving as “engaging in a task for which the solution method is not known in advance” (NCTM, 2000, p. 52) and declare that problem solving is an “integral part of all mathematics learning, not an isolated part of the mathematics program. It should be a well-integrated part of the curriculum that supports the development of mathematical understanding” (NCTM, 2000, p. 52).

The *Principles and Standards* emphasizes that problem solving should contribute to mathematics content knowledge, contexts should include areas other than mathematics, students should be able to apply strategies, and students should have metacognitive knowledge of problem solving (NCTM, 2000, p. 52). This is one theme out of multiple themes that run throughout the *Principles and Standards* in a consistent manner in all the grade levels.

From pre-kindergarten to 2nd grade, it is stated that problem solving “is finding a way to reach a goal that is not immediately attainable” (NCTM, 2000, p. 116). From 3rd to 5th grade, we see a slightly expanded view of problem solving as the authors state that “students who can efficiently and accurately multiply but who cannot identify situations that call for multiplication are not well prepared” (NCTM, 2000, p. 182). The definition of problem solving implied by this sentence does not indicate that the situation must be nonroutine for the student. But, for the remaining grade levels, the authors return to the idea of problem solving as involving problems that are nonroutine to the solver.

Mathematics Educators' Definition of Problem Solving

The definitions given by the mathematics educators were consistent with NCTM's definition. In particular, every mathematics educator placed an emphasis on the necessity of problems being of a nonroutine nature. As long as the solution path is unfamiliar to the student, then whatever the student does to reach a solution is considered problem solving.

Most of the answers stopped with that requirement. Individual mathematics educators (no greater than one on each of the following) included some additional criteria beyond the nonroutine nature of the problem. These additional criteria are: the solution path should be within reach of the student, there must be a "want or need to find a solution," reasoning must be used in the process of solving, and knowledge must be used in the process of solving.

An interesting note is that none of the mathematics educators required that the student achieve some level of quality (however quality is defined) in order to consider their process problem solving. Additional requirements listed included the possibility of reaching a solution; motivation; a process of reasoning; and use of existing information. It might be argued that these additional requirements create a certain level of quality. But, these requirements themselves do not distinguish between a "good" problem solver and a "poor" one, only between a problem solver and a non-problem solver. Again, it might be argued that a good problem solver is closer to the possibility of reaching a solution, has more motivation, uses more reasoning, or has more knowledge than a poor problem solver. Caution should be taken in this interpretation, because there is really nothing in the mathematics educators' definitions to assume this distinction.

In summary, from the mathematics educators' viewpoint, problem solving occurs in the process that the student goes through in order to resolve a situation for which the method of approach is unfamiliar or not obvious (nonroutine) to the student. Beyond this, there is no general agreement.

Mathematicians' Definition of Problem Solving

The mathematicians differed from the mathematics educators in that no one mentioned that the problem must be nonroutine. All of the mathematicians put some requirement on the process that seemed to eliminate the solving of textbook problems in which the student is told what method to use as problem solving.

Further, many of the mathematicians required a certain depth to the process in order to consider it problem solving. For example, one mathematician required that the student be focussed and making constructive progress toward a solution. Another required that problem solving "include the idea of development and implementation of some kind of strategy."

Some of the mathematicians attempted to make problem solving into a more detailed process than the mathematics educators. For example, one mathematician defined problem solving to be "the process of evaluating possible techniques, applying techniques, reaching a solution, checking the results for accuracy, and writing out the solution in a coherent fashion." Polya's problem-solving process (1945/1973) is similar. Research has shown that often students do not review their work once they feel a problem is solved (Schoen & Oehmke, 1980; Singh, 1982). It is possible that this mathematician would not call that process problem solving, while mathematics educators might. Certainly the mathematics educator would still prefer that the students review their work when they have solved problems. Another stated that "problem solving is engaging in a task for which the solution method requires more than a one step process and the process requires some choices along the way." In general, the mathematicians attempted to describe an algorithmic approach.

In summary, from the mathematicians' perspectives, problem solving is what a student does when he or she is given a task and not *told* how to approach it (even though he or she may very well immediately know what steps to take).

<u>Conclusion And Implications</u>

This paper has presented three points of view of what is problem solving. While the *Principles and Standards* and mathematics educators are in close agreement, there is a subtle difference that occurs in the views of mathematicians. There appear to be two types of problem solving with the key difference residing in whether the problem contains the element of nonroutine. Mathematics educators and the *Principles and Standards* emphasize that problem solving occurs only when the problem is nonroutine to the solver (the method is not immediately obvious). The mathematicians in this study do not consider if the solver views the problem as routine or knows the method immediately, as long as the method is not specified in the problem. This difference in definition may indeed be subtle but it creates several dilemmas.

Many undergraduate mathematics departments struggle with how to accurately place incoming students into mathematics courses with use of a mathematics placement exam. But, what should that exam test? It might be true that mathematics professors would value students arriving with considerable mathematics knowledge and mathematics skills (even if these are routine skills) as much as them arriving with the ability to solve nonroutine problems. But, a problem-solving test would look quite different if it tested mathematics knowledge and mathematics skills versus the ability to solve nonroutine problems.

Another dilemma is the issue of just what constitutes nonroutine, even if the mathematical community includes nonroutine in the definition of problem solving. Of course, nonroutine would apply to the test taker, and in that sense may not be a quality that resides in a test itself. This makes it very difficult to construct a problem-solving test. Even without the problem of what is routine and nonroutine, there is not agreement on what problem solving is, and, therefore, it is nearly impossible to test problem solving. When writing standardized mathematics tests, should the tests consist of "routine" problems in which a student is required to apply unspecified methods? Again, a mathematician might be pleased that a student did well on this test. Or should the test have an open-ended format with nonroutine problems?

Returning to the placement test that is appropriate to place incoming students into mathematics courses, many mathematicians may feel that the ability to solve nonroutine problems will result through the process of undergraduate study and consider it more important that students arrive with an ability to solve routine problems.

Mathematicians and mathematics educators are not mutually exclusive, and the point of all of this is not to divide them into two groups. Problem solving at this point in time, at least, lies in the "eye of the beholder" and it is important to understand that there is not just one definition of problem solving in action but many. Yet, definitions are an important part of mathematics and one might suggest that as occurs in mathematics, sometimes a definition simply must be decided on and used. Perhaps there needs to be a separate definition or term used for problem solving in a nonroutine situation.

As it stands, the mathematical community is using one term (problem solving) to mean possibly two different things. Further, if the definition of problem solving only includes nonroutine situations, there is an entire set of problem-solving skills that are being overlooked. In the past when NCTM has suggested decreased emphasis on, for instance, procedural skills, some took that to mean no emphasis (NCTM, 1989). *Principles and Standards* attempted to correct that by including and valuing procedural skills. For example, the "learning principle" includes several references to procedures clarifying that it is important, but that understanding must accompany the process of learning procedures, not just memorizing procedures (2000, pp. 20-21).

There is a similar situation with problem solving. Problem solving of nonroutine problems is an important skill. Problem solving of routine problems is also a necessary skill and valued by mathematicians. Problem solving of routine problems is not simply applying procedural skills, because again there is the step where the student must decide which procedure to apply. In attempting to answer the question "what is problem-solving ability?" we need to decide what it is that we value and perhaps it is better to value more than less. Perhaps we should be as inclusive as possible in our definition of problem solving, or failing that, allow for two definitions of problem solving. Regardless of how this is handled, mathematicians value the problem solving of routine skills, and this is an area of mathematics that should not be ignored or devalued in the education of secondary mathematics students.

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